Fractional Orlicz-Sobolev spaces

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References of papers

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[2] **A. A., A. Cianchi, L. Pick, L. Slavíková**, **On the limit as $s \to 1^-$ of possibly non-separable fractional Orlicz-Sobolev spaces**, *Atti Accad. Naz. Lincei, Rend. Lincei Mat. Appl.*, to appear.

[3] **A. A., A. Cianchi, L. Pick, L. Slavíková**, **On the limit as $s \to 0^+$ of fractional Orlicz-Sobolev spaces**, *J. Fourier Anal. Appl.*, **26** (2020).

Classical Fractional Sobolev embeddings

Let $s \in (0,1)$, $p \in [1,\infty)$. The **fractional Sobolev space** $W^{s,p}(\mathbb{R}^n)$ is defined as

$$W^{s,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : |u|_{s,p,\mathbb{R}^n} < \infty\},$$

where

$$|u|_{s,p,\mathbb{R}^n} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{|u(x)-u(y)|}{|x-y|^s} \right)^p \frac{dx\,dy}{|x-y|^n} \right)^{\frac{1}{p}}$$

is the Gagliardo-Slobodeckij seminorm.

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Classical fractional Sobolev type embedding

Let $s \in (0,1)$. If $1 \le p < \frac{n}{s}$, then $\exists C$ s.t.

$$\|u\|_{L^{\frac{np}{n-sp}}(\mathbb{R}^n)} \le C\,|u|_{s,p,\mathbb{R}^n}$$

for every measurable u decaying to 0 near infinity.

Let $A : [0, \infty) \to [0, \infty]$ be a Young function, namely

a convex function s.t. $A(0) = 0$.

The **Orlicz space** $L^A(\mathbb{R}^n)$ is a Banach space equipped with the norm

$$\|u\|_{L^A(\mathbb{R}^n)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} A\left(\frac{|u|}{\lambda}\right) dx \leq 1\right\}.$$

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$$W^{s,A}(\mathbb{R}^n) = \{u \in L^A(\mathbb{R}^n) : |u|_{s,A,\mathbb{R}^n} < \infty\},$$

where

$$|u|_{s,A,\mathbb{R}^n} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x) - u(y)|}{\lambda |x - y|^s}\right) \frac{dx\, dy}{|x - y|^n} \leq 1 \right\}$$

is a seminorm of u.

Set

$$\mathcal{M}(\mathbb{R}^n) = \{u : \mathbb{R}^n \to \mathbb{R} : u \text{ is measurable}\}$$

and

$$\mathcal{M}_d(\mathbb{R}^n) = \Big\{u \in \mathcal{M}(\mathbb{R}^n) : |\{x \in \mathbb{R}^n : |u(x)| > t\}| < \infty \text{ for every } t > 0\Big\}.$$

Namely, the subset of $\mathcal{M}(\mathbb{R}^n)$ of those functions u decaying near infinity.

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The homogeneous fractional Orlicz-Sobolev space $V^{s,A}(\mathbb{R}^n)$ is

$$V^{s,A}(\mathbb{R}^n) = \{u \in \mathcal{M}(\mathbb{R}^n) : |u|_{s,A,\mathbb{R}^n} < \infty\}.$$

Set

$$V_d^{s,A}(\mathbb{R}^n) = V^{s,A}(\mathbb{R}^n) \cap \mathcal{M}_d(\mathbb{R}^n).$$

Optimal fractional Orlicz target space

Pb: Optimal fractional Orlicz-Sobolev embeddings?

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Let A be a Young function such that

$$\int^{\infty}\left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}}dt=\infty \quad \text{and} \quad \int_0\left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}}dt<\infty. \qquad (1)$$

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- If $A(t) = t^p$, the first condition in (1) corresponds to assumption

$$1 \leq p \leq \frac{n}{s}.$$

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Define the function $H : [0, \infty) \to [0, \infty)$ as

$$H(t) = \left(\int_0^t \left(\frac{\tau}{A(\tau)}\right)^{\frac{s}{n-s}} d\tau\right)^{\frac{n-s}{n}} \quad \text{for } t \geq 0$$

and the Young function $A_{\frac{n}{s}}$ by

$$A_{\frac{n}{s}}(t) = A(H^{-1}(t)) \quad \text{for } t \geq 0. \qquad (2)$$

- The function $A_{\frac{n}{s}}$ is the optimal fractional Sobolev conjugate of A.

Optimal fractional Orlicz target space

Theorem 1: Optimal fractional Orlicz target space [ACPS 1]

Let $s \in (0,1)$. Let A be a Young function fulfilling conditions in (1).
Let $A_{\frac{n}{s}}$ be the Young function defined as in (2).
Then,

$$V_d^{s,A}(\mathbb{R}^n) \to L^{A_{\frac{n}{s}}}(\mathbb{R}^n),$$

and $\exists\, C$ s.t.

$$\|u\|_{L^{A_{\frac{n}{s}}}(\mathbb{R}^n)} \le C|u|_{s,A,\mathbb{R}^n} \qquad \forall\, u \in V_d^{s,A}(\mathbb{R}^n). \tag{3}$$

Moreover, $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ is **the optimal (the smallest) fractional** target space in inequality (3) **among all Orlicz spaces.**

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Remark: Condition $\displaystyle\int_0 \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt < \infty$ is necessary for the embedding of $V_d^{s,A}(\mathbb{R}^n)$ into $L^B(\mathbb{R}^n)$, for any Young function B.

Remark

Remark 1: Even though setting $s = 1$ in the definition of the fractional Orlicz-Sobolev space $W^{s,A}(\mathbb{R}^n)$ does not recover the first-order Orlicz-Sobolev space $W^{1,A}(\mathbb{R}^n)$, setting $s = 1$ in the definition of $A_{\frac{n}{s}}$ one recovers the optimal Sobolev conjugate A_n of A for $W^{1,A}(\mathbb{R}^n)$ discovered in **[Cianchi 1996, 1997]**, namely

$$V_d^{1,A}(\mathbb{R}^n) \to L^{A_n}(\mathbb{R}^n)\,,$$

where

$$A_n(t) = A(H^{-1}(t)) \quad \text{for } t \geq 0,$$

with

$$H(t) = \left(\int_0^t \left(\frac{\tau}{A(\tau)} \right)^{\frac{1}{n-1}} d\tau \right)^{\frac{n-1}{n}} \quad \text{for } t \geq 0\,.$$

Moreover, $L^{A_n}(\mathbb{R}^n)$ is the **optimal Orlicz** target space.

Example 1: Power Young functions

Example 1: Let A be a Young function defined as

$$A(t) \approx t^p \quad \text{near infinity,}$$

with $1 \leq p \leq \frac{n}{s}$. **Theorem 1** recovers the classical fractional Sobolev embedding

$$V_d^{s,p}(\mathbb{R}^n) \to L^{A_{\frac{n}{s}}}(\mathbb{R}^n),$$

and

$$\|u\|_{L^{A_{\frac{n}{s}}}(\mathbb{R}^n)} \leq C|u|_{s,p,\mathbb{R}^n} \qquad \forall\, u \in V_d^{s,p}(\mathbb{R}^n),$$

where

$$A_{\frac{n}{s}}(t) \approx \begin{cases} t^{\frac{np}{n-sp}} & \text{if } 1 \leq p < \dfrac{n}{s} \\ e^{t^{\frac{n}{n-s}}} & \text{if } p = \dfrac{n}{s} \end{cases} \qquad \text{near infinity.}$$

Moreover, $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ is the **optimal Orlicz** target space.

Example 2: Let A be a Young function defined as

$$A(t) \approx t^p (\log t)^\alpha \quad \text{near infinity,}$$

where either $1 \le p < \frac{n}{s}$ and $\alpha \in \mathbb{R}$, or $p = \frac{n}{s}$ and $\alpha \le \frac{n}{s} - 1$.
Then, **Theorem 1** tells us that

$$V_d^{s,A}(\mathbb{R}^n) \to L^{A_{\frac{n}{s}}}(\mathbb{R}^n)\,,$$

where

$$A_{\frac{n}{s}}(t) \approx \begin{cases} t^{\frac{np}{n-sp}}(\log t)^{\frac{\alpha n}{n-sp}} & \text{if } 1 \le p < \dfrac{n}{s} \\ e^{t^{\frac{n}{n-(\alpha+1)s}}} & \text{if } p = \dfrac{n}{s} \text{ and } \alpha < \dfrac{n}{s} - 1 \\ e^{e^{t^{\frac{n}{n-s}}}} & \text{if } p = \dfrac{n}{s} \text{ and } \alpha = \dfrac{n}{s} - 1 \end{cases} \quad \text{near infinity.}$$

Moreover, $L^{A_{\frac{n}{s}}}(\mathbb{R}^n)$ is the **optimal Orlicz** target space.

Rearrangement-invariant spaces

A Banach function space $X(\mathbb{R}^n)$ is said a **rearrangement-invariant** (briefly, **r.i.**) **space** if

$$\|u\|_{X(\mathbb{R}^n)} = \|v\|_{X(\mathbb{R}^n)} \qquad \text{if} \quad u^* = v^* \,.$$

In a sense, the norm $\|u\|_{X(\mathbb{R}^n)}$ only depends on global integrability properties of u over $\mathbb{R}^n$.

Here, u^* denotes the decreasing rearrangement of u.

- **Examples:**
- Lebesgue spaces $L^p(\mathbb{R}^n)$
- Orlicz spaces $L^A(\mathbb{R}^n)$
- Lorentz spaces $L^{p,q}(\mathbb{R}^n)$

Improved inequality

Optimal r.i. space for classical fractional Sobolev type embedding

Let $s \in (0,1)$. If $1 \le p < \frac{n}{s}$, then $\exists C$ s.t.

$$\|u\|_{L^{\frac{np}{n-sp},p}(\mathbb{R}^n)} \le C\,|u|_{s,p,\mathbb{R}^n}$$

for every measurable u decaying to 0 near infinity.
Moreover, $L^{\frac{np}{n-sp},p}(\mathbb{R}^n)$ is the optimal r.i. target space .

(See **[Frank-Seiringer, 2008]**)

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- The Lorentz space $L^{\frac{np}{n-sp},p}(\mathbb{R}^n)$ is equipped with the norm

$$\|u\|_{L^{\frac{np}{n-sp},p}(\mathbb{R}^n)} = \left\| r^{-\frac{s}{n}} u^*(r) \right\|_{L^p(0,\infty)} .$$

- $L^{\frac{np}{n-sp},p}(\mathbb{R}^n) \not\subseteq L^{\frac{np}{n-sp}}(\mathbb{R}^n)$

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Pb: Analogous improvement for fractional Orlicz-Sobolev spaces?

Optimal r.i. target for $W^{s,A}(\mathbb{R}^n)$

Let $s \in (0,1)$. Let A be a Young function fulfilling conditions in (1). Define the Young function $\widehat{A}$ as

$$\widehat{A}(t) = \int_0^t \widehat{a}(r)\,dr \quad \text{for } t \geq 0, \tag{4}$$

where

$$\widehat{a}^{\,-1}(r) = \left(\int_{a^{-1}(r)}^{\infty} \left(\int_0^t \left(\frac{1}{a(\varrho)} \right)^{\frac{s}{n-s}} d\varrho \right)^{-\frac{n}{s}} \frac{dt}{a(t)^{\frac{n}{n-s}}} \right)^{\frac{s}{s-n}} \quad \text{for } r \geq 0.$$

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- $\widehat{A}(t) \lesssim A(t)$ near infinity.
- $\widehat{A}(t) \approx A(t)$ near infinity, if $A(t)$ "$<<$" $t^{\frac{n}{s}}$ in a suitable sense.
- If $A(t) = t^p$ and $1 \leq p < \frac{n}{s} \implies \widehat{A}(t) \approx t^p$ near infinity.
- If $A(t) = t^{\frac{n}{s}} \implies \widehat{A}(t) \approx \left(\frac{t}{\log t} \right)^{\frac{n}{s}}$ near infinity.

Optimal r.i. target space for $W^{s,A}(\mathbb{R}^n)$

Let $L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)$ be the Orlicz-Lorentz space equipped with the norm

$$\|u\|_{L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)} = \|r^{-\frac{s}{n}} u^*(r)\|_{L^{\widehat{A}}(0,\infty)}\,.$$

Optimal r.i. target space for $W^{s,A}(\mathbb{R}^n)$

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Theorem 2: Optimal r.i. target space [ACPS 1]

Let $s \in (0,1)$. Let A be a Young function as in **Theorem 1**.
Let $\widehat{A}$ be the Young function defined as in (4).
Then,

$$V_d^{s,A}(\mathbb{R}^n) \to L(\widehat{A}, \tfrac{n}{s})(\mathbb{R}^n),$$

and $\exists\, C$ s.t.

$$\|u\|_{L(\widehat{A},\frac{n}{s})(\mathbb{R}^n)} \le C|u|_{s,A,\mathbb{R}^n}$$

for every function $u \in V_d^{s,A}(\mathbb{R}^n)$.
Moreover, $L(\widehat{A}, \frac{n}{s})(\mathbb{R}^n)$ is the **optimal r.i.** target space.

A crucial step in our approach to fractional Orlicz-Sobolev embeddings is a **fractional Orlicz-Hardy type inequality** of order $s \in (0,1)$.

Recall the **classical Hardy inequality** in $W^{1,p}(\mathbb{R}^n)$:

Theorem: Hardy inequality in $W^{1,p}(\mathbb{R}^n)$

Let $1 \leq p < n$. Assume that $\nabla u \in L^p(\mathbb{R}^n)$.
Then, there exists a constant C s.t.

$$\left\| \frac{|u(x)|}{|x|} \right\|_{L^p(\mathbb{R}^n)} \leq C \, \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

for every $u \in \mathcal{M}_d(\mathbb{R}^n)$.

Fractional Orlicz-Hardy inequality

Pb: Fractional Orlicz-Hardy type inequality of order $s \in (0,1)$?

Fractional Orlicz-Hardy inequality

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Theorem 3: Fractional Orlicz-Hardy inequality [ACPS 1]

Let $s \in (0,1)$. Let A be a Young function as in **Theorem 1**.
Let $\widehat{A}$ be the Young function defined by (4).
Then, there exists a constant C s.t.

$$\left\| \frac{|u(x)|}{|x|^s} \right\|_{L^{\widehat{A}}(\mathbb{R}^n)} \leq C\, |u|_{s,A,\mathbb{R}^n} \qquad \forall u \in V_d^{s,A}(\mathbb{R}^n).$$

Moreover,

$$\int_{\mathbb{R}^n} \widehat{A}\left(\frac{|u(x)|}{|x|^s}\right) dx \leq (1-s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(C\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{dx\,dy}{|x-y|^n}$$

for every $u \in \mathcal{M}_d(\mathbb{R}^n)$.

Remark

Remark 2: In contrast with the classical and fractional Sobolev cases corresponding to the case when $A(t) = t^p$, namely with $W^{1,p}(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n)$, respectively, the Young function $\widehat{A}$ is not always equivalent to A.

This is due to the generality of conditions (1), namely

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt = \infty \quad \text{and} \quad \int_0 \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt < \infty$$

which extend the assumption $1 \leq p < \frac{n}{s}$ required in the classical case, and allow for Young functions $A(t)$ whose growth can be very close to that of the critical power $t^{\frac{n}{s}}$.

Pb: The fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ does not agree with the classical integer-order Sobolev spaces when the order of smoothness s is formally set to an integer

When s tends to an integer...

Pb: The fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ does not agree with the classical integer-order Sobolev spaces when the order of smoothness s is formally set to an integer

- However, some twenty years ago, it was discovered that

a suitably **normalized** Gagliardo-Slobodeckij seminorm in $W^{s,p}(\mathbb{R}^n)$ recovers, in the **limit** as $s \to 1^-$ or $s \to 0^+$, its integer-order counterpart.

The result was first established at the endpoint 1^-.

Theorem A [Bourgain-Brezis-Mironescu, 2001, 2002]

Let $1 \leq p < \infty$. If

$$u \in W^{1,p}(\mathbb{R}^n)\,,$$

then

$$\lim_{s\to 1^-} (1-s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)^p \frac{dx\,dy}{|x-y|^n} = K(p,n)\int_{\mathbb{R}^n} |\nabla u|^p\,dx.$$

Here, $$K(p,n) = \frac{1}{p}\int_{\mathbb{S}^{n-1}} |\theta\cdot e|^p \, d\mathcal{H}^{n-1}(\theta)\,,$$

- $\mathbb{S}^{n-1}$ denotes the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$
- $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure
- e is any point on $\mathbb{S}^{n-1}$.

A **converse** of **Theorem A** holds if $1 < p < \infty$:

Theorem B [Bourgain-Brezis-Mironescu, 2001, 2002]

Let $1 < p < \infty$ and let $u \in L^p(\mathbb{R}^n)$.
If

$$\liminf_{s \to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{|u(x) - u(y)|}{|x-y|^s} \right)^p \frac{dx\, dy}{|x-y|^n} < \infty ,$$

then

$$u \in W^{1,p}(\mathbb{R}^n).$$

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then

$$u \in W^{1,p}(\mathbb{R}^n).$$

If

$$p = 1 ,$$

the latter result can **fail**!

A version of **Theorem A** also holds in $BV(\mathbb{R}^n)$.

Theorem C [Bourgain-Brezis-Mironescu, 2001, 2002]

If

$$u \in BV(\mathbb{R}^n),$$

then

$$\lim_{s\to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|^s} \frac{dx\,dy}{|x-y|^n} = K(1,n)\,\|Du\|(\mathbb{R}^n),$$

where $\|Du\|(\mathbb{R}^n)$ denotes the total variation of the measure Du.

$s \to 1^-$

Theorem B admits a **counterpart** where $W^{1,1}(\mathbb{R}^n)$ is replaced by $BV(\mathbb{R}^n)$.

A slight variant of the assertions above is contained in

- **[Van Schaftingen-Willem, 2004]**

Theorem D [Van Schaftingen-Willem, 2004]

Let $u \in L^1(\mathbb{R}^n)$. If

$$\liminf_{s \to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|^s} \frac{dx\,dy}{|x-y|^n} < \infty,$$

then

$$u \in BV(\mathbb{R}^n).$$

Pb: What happens in the more general setting of $W^{s,A}(\mathbb{R}^n)$?

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Define the Young function $A_\circ$ as

$$A_\circ(t) = \int_0^t \int_{\mathbb{S}^{n-1}} A(r\,|\theta \cdot e|)\, d\mathcal{H}^{n-1}(\theta)\, \frac{dr}{r} \qquad \text{for } t \geq 0\,, \tag{5}$$

where e is any fixed unit vector in $\mathbb{S}^{n-1}$.

- The right-hand side of (5) is independent of the choice of e.
- $A_\circ$ is always **equivalent** to A, namely $\exists\, c_1, c_2$ s.t.

$$A(c_1 t) \leq A_\circ(t) \leq c_2 A(t) \quad \text{for } t \geq 0.$$

Theorem 4 [ACPS 2]

Let $s \in (0,1)$ and let A be a finite-valued Young function.
If

$$u \in W^{1,A}(\mathbb{R}^n),$$

then, there exists $\lambda_0 > 0$ such that

$$\lim_{s \to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{\lambda |x-y|^s} \right) \frac{dx\, dy}{|x-y|^n} = \int_{\mathbb{R}^n} A_\circ \left(\frac{|\nabla u|}{\lambda} \right) dx$$

for every $\lambda \geq \lambda_0$.

Remarks

Remark 3: Property in **Theorem 4** holds for u/λ, with sufficiently large λ, and not for u itself because if $A \notin \Delta_2$, then the fact that $\nabla u \in L^A(\mathbb{R}^n)$ only ensures that $\displaystyle\int_{\mathbb{R}^n} A\left(\frac{|\nabla u|}{\lambda}\right) dx < \infty$ and hence $\displaystyle\int_{\mathbb{R}^n} A_\circ\left(\frac{|\nabla u|}{\lambda}\right) dx < \infty$, for sufficiently large λ.

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Remark 5: In particular, **Theorem 4** improves results in **[Fernandez Boder-Salort, 2019]** that only allow for $A \in \Delta_2$ and require an extra unnecessary hypothesis, and $A_\circ$ appears in less explicit form.

In the light of the restriction $p > 1$ in **Theorem B** by **[Bourgain-Brezis-Mironescu, 2001, 2002]** to imply $u \in W^{1,p}(\mathbb{R}^n)$, a converse to **Theorem 4** requires some additional assumptions on A:

$$\lim_{t\to\infty} \frac{A(t)}{t} = \infty, \quad \text{superlinear growth near infinity} \tag{6}$$

and

$$\lim_{t\to 0^+} \frac{A(t)}{t} = 0 \quad \text{sublinear decay at } 0. \tag{7}$$

A finite-valued Young function fulfilling (6) and (7) is called N-**function**.

A converse to Theorem 4

Theorem 5 [ACPS 2]

Let $s \in (0,1)$ and let A be a N-function .
If $u \in L^A(\mathbb{R}^n)$ and $\exists\, \lambda > 0$ s.t.

$$\liminf_{s\to 1^-} (1-s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{\lambda |x-y|^s}\right) \frac{dx\,dy}{|x-y|^n} < \infty,$$

then $u \in W^{1,A}(\mathbb{R}^n)$.

Notations for next theorem

Let $u \in BV(\mathbb{R}^n)$.

- ∇u denotes the absolutely continuous part of the measure Du with respect to the Lebesgue measure.
- $D^s u$ denotes singular part of Du.
- $\|D^s u\|(\mathbb{R}^n)$ denotes the total variation of $D^s u$ over $\mathbb{R}^n$.
- $\int_{\mathbb{R}^n} A_\circ(|\nabla u|)\,dx + a_\circ^\infty \|D^s u\|(\mathbb{R}^n)$ denotes the relaxed functional of $\int_{\mathbb{R}^n} A_\circ(|\nabla u|)\,dx$ with respect to convergence in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.
- The number $a_\circ^\infty$ is defined as

$$a_\circ^\infty = \lim_{t \to \infty} \frac{A_\circ(t)}{t}$$

(see **[Goffman-Serrin, 1964]**).

- $a_\circ^\infty < \infty$ thanks to $A_\circ \approx A$ due to assumption (8) below.

Counterpart of Theorem 4

In the case when A has a linear growth near infinity or near 0, **Theorem 4** and **Theorem 5**, respectively, have counterparts in the framework of BV-functions.

Theorem 6 (Counterpart of Theorem 4) [ACPS 2]

Let $s \in (0,1)$ and let A be a Young function s.t.

$$\lim_{t\to\infty} \frac{A(t)}{t} < \infty \quad \text{linear growth near infinity.} \tag{8}$$

Assume that $u \in BV(\mathbb{R}^n)$. Then,

$$\lim_{s\to 1^-} (1-s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{dx\,dy}{|x-y|^n}$$

$$= \int_{\mathbb{R}^n} A_\circ\left(|\nabla u|\right) dx + a_\circ^\infty \|D^s u\|(\mathbb{R}^n).$$

Counterpart of Theorem 5

Theorem 7 (Counterpart of Theorem 5) [ACPS 2]

Let $s \in (0,1)$ and let A be a finite-valued Young function s.t.

$$\lim_{t \to 0^+} \frac{A(t)}{t} > 0 \quad \text{linear growth near zero.}$$

Assume that $u \in L^1(\mathbb{R}^n)$ is s.t.

$$\liminf_{s \to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{\lambda |x-y|^s} \right) \frac{dx\, dy}{|x-y|^n} < \infty$$

for some $\lambda > 0$. Then, $u \in BV(\mathbb{R}^n)$.

A suitably **normalized** Gagliardo-Slobodeckij seminorm in $W^{s,p}(\mathbb{R}^n)$ recovers, in the **limit** as $s \to 0^+$, its integer-order counterpart.
This problem was solved in **[Maz'ya-Shaposhnikova, 2002]**.

Theorem [Maz'ya-Shaposhnikova, 2002]

Let $p \geq 1$. Then,

$$\lim_{s\to 0^+} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)^p \frac{dx\,dy}{|x-y|^n} = \frac{2\,n\omega_n}{p}\int_{\mathbb{R}^n} |u(x)|^p\,dx$$

for every function u decaying to 0 near infinity and making the double integral finite for some $s \in (0,1)$,

Pb: What happens in the more general setting of $W^{s,A}(\mathbb{R}^n)$?

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- Define the Young function $\overline{A}$ associated with A by

$$\overline{A}(t) = \int_0^t \frac{A(\tau)}{\tau}\, d\tau \qquad \text{for } t \geq 0\,.$$

- $A \approx \overline{A}$ since $A(t/2) \leq \overline{A}(t) \leq A(t)$ for $t \geq 0$.

Theorem 8 [ACPS 3]

Let $A \in \Delta_2$. Assume that $u \in \bigcup_{s\in(0,1)} V_d^{s,A}(\mathbb{R}^n)$. Then

$$\lim_{s\to 0^+} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n} = 2\,n\omega_n \int_{\mathbb{R}^n} \overline{A}(|u(x)|)\,dx\,. \tag{9}$$

Remark 6: Plainly, **Theorem 8** recovers **Theorem** in **[Maz'ya-Shaposhnikova, 2002]** when $A(t) = t^p$ for some $p \geq 1$, since $\overline{A}(t) = \frac{t^p}{p}$ in this case.

$s \to 0^+$

Remark 7: A partial result in this connection is contained in the recent contribution **[Capolli-Maione-Salort-Vecchi, Preprint]**, where bounds for

$$\liminf_{s\to 0^+} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n}$$

and

$$\limsup_{s\to 0^+} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right)\frac{dx\,dy}{|x-y|^n}$$

are given for Young functions A satisfying the Δ_2-condition.

Our results provide a full answer to the relevant problem.
We prove that, under the Δ_2-condition on A, the limit in (9) does exist, and equals the integral of a function of $|u|$ over $\mathbb{R}^n$.
Moreover, we show that the result can fail if the Δ_2-condition is dropped.

Indispensability of Δ_2-condition

The indispensability of the Δ_2-condition for the function A is demonstrated via the next result.

Theorem 9 [ACPS 3]

There exist Young functions $A \notin \Delta_2$, and corresponding functions $u : \mathbb{R}^n \to \mathbb{R}$ such that $u \in V_d^{s,A}(\mathbb{R}^n)$ for every $s \in (0,1)$,

$$\int_{\mathbb{R}^n} \overline{A}(|u(x)|)\, dx \le \int_{\mathbb{R}^n} A(|u(x)|)\, dx < \infty\,,$$

but

$$\lim_{s\to 0^+} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A\left(\frac{|u(x)-u(y)|}{|x-y|^s}\right) \frac{dx\,dy}{|x-y|^n} = \infty\,.$$

ADDITIONAL RESULTS

- **Higher-order fractional Orlicz-Sobolev inequalities with $s \in (0, n) \setminus \mathbb{N}$.**
- $\Omega \subseteq \mathbb{R}^n$ **open bounded.**
- **Compact embeddings.**

Dziękuję bardzo!!

Many thanks!!