

A Complexity Dichotomy for Semilinear Target Sets in Automata with One Counter

Henry Sinclair-Banks

Based on joint work with Yousef Shakiba and Georg Zetsche



Automata in the Wild 2025

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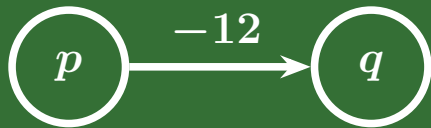
University of Warwick, UK

“Automata with One Counter” is a strange turn of phrase...

Integer Semantics

Transitions have updates in \mathbb{Z} .

The counter value is in \mathbb{Z} .

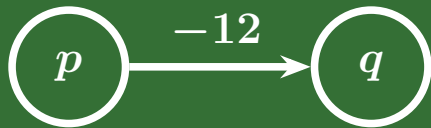


$$p(5) \rightarrow q(-7)$$

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Natural Semantics

Transitions have updates in \mathbb{N} .

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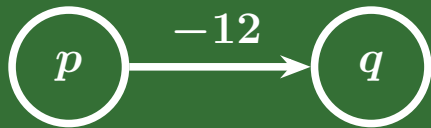


$$p(5) \rightarrow q(17)$$

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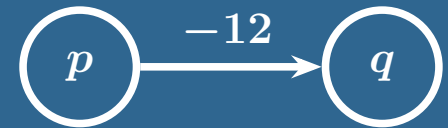


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VASS Semantics

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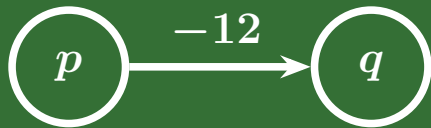


$$p(15) \rightarrow q(3)$$

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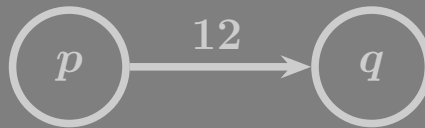


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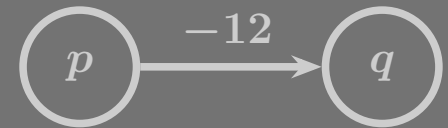


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Motivation

Reachability. From $p(x)$, can you reach $q(y)$?

Coverability. From $p(x)$, can you reach $q(y')$ such that $y' \geq y$?

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Theorem. Reachability (with integer semantics) is NP-hard.

Theorem. Coverability (with integer semantics) is in $AC^1 \subseteq P$.

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Theorem. Reachability (with integer semantics) is **HARD**

Theorem. Coverability (with integer semantics) is **EASY**

What makes reachability hard and coverability easy?

Generalising Reachability and Coverability

φ -REACHABILITY

Fixed: A Presburger formula φ (with $p + 1$ variables).

Input: An integer one-counter automaton \mathcal{A} , an initial configuration $p(x)$, and a vector $\vec{t} \in \mathbb{Z}^p$.

Question: Does there exist a reachable configuration $q(y)$ such that $\varphi(\vec{t}, y)$ holds?

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Examples

Reachability: $\varphi(t, y) \Leftrightarrow y = t$.

Coverability: $\varphi(t, y) \Leftrightarrow y \geq t$.

Cover and avoid: $\varphi(t_1, t_2, y) \Leftrightarrow (y \geq t_1) \wedge (y \neq t_2)$.

Reach an interval: $\varphi(t, y) \Leftrightarrow y \in [t, 2t]$.

Reach an interval or a negative value: $\varphi(t, y) \Leftrightarrow y \in (-\infty, 0] \cup [t, 2t]$.

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Main Contribution

Theorem. Let φ be a Presburger formula with $p + 1$ variables.

(1) If φ is **dense**, then φ -REACHABILITY is in AC^1 .

(2) Otherwise, φ -REACHABILITY is NP-hard.

Density^{**}: Let $S \subseteq \mathbb{Z}$ be a semilinear set... $Density(S) = \inf_{y \in S} \inf_{n \in \mathbb{N}} \frac{|S \cap [y-n, y+n]|}{2n+1}$

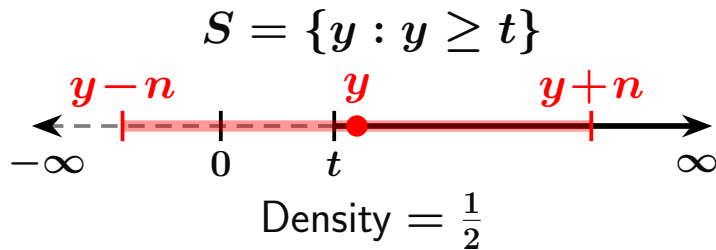
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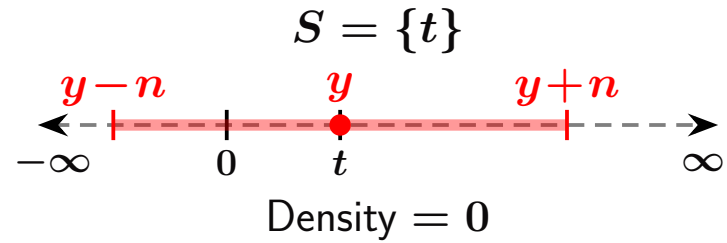
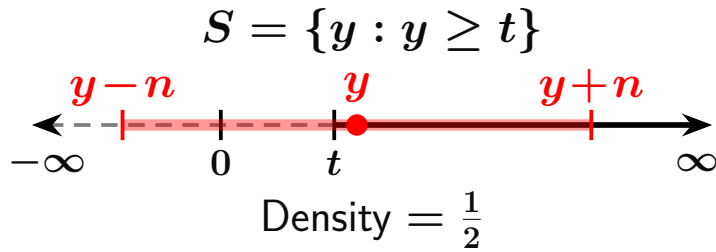
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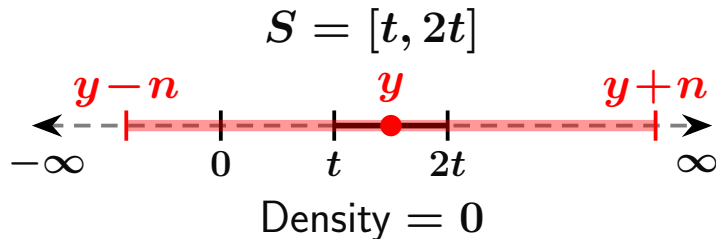
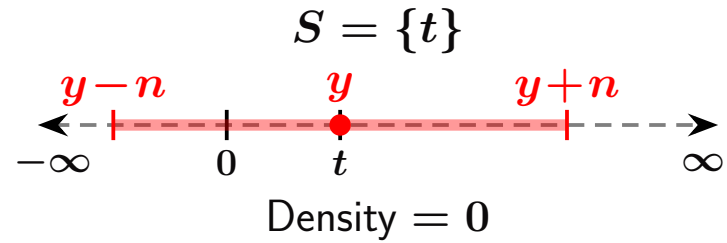
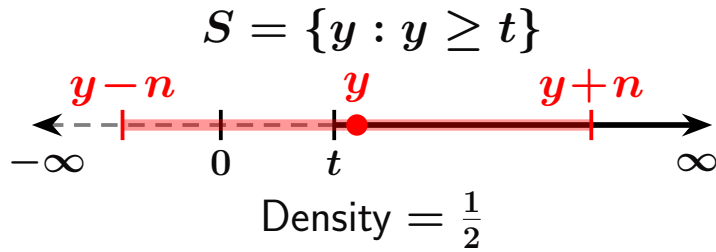
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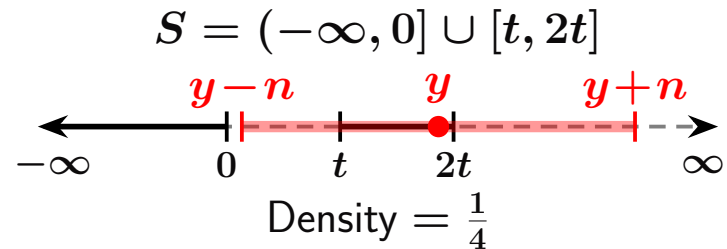
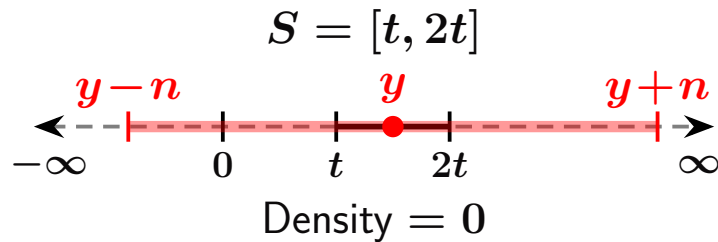
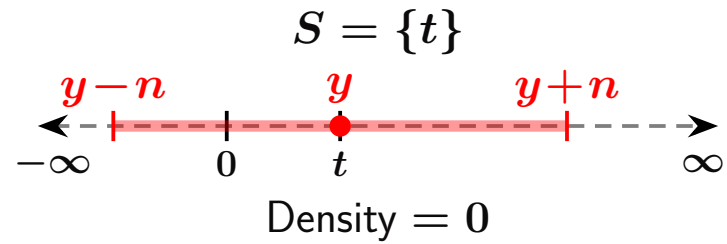
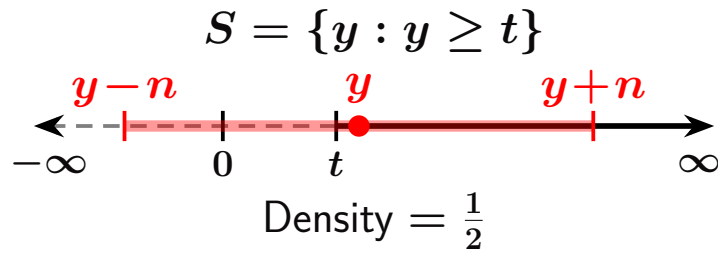
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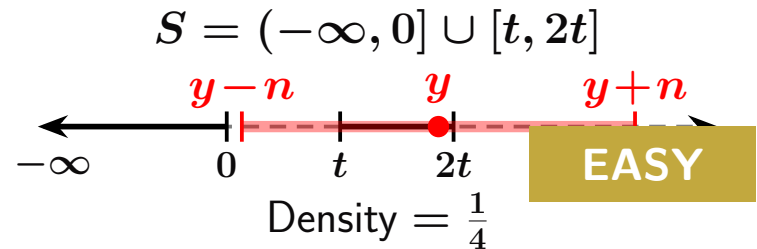
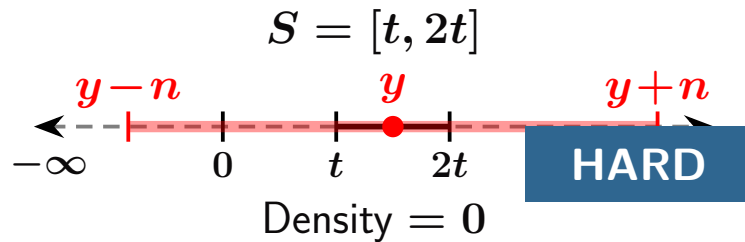
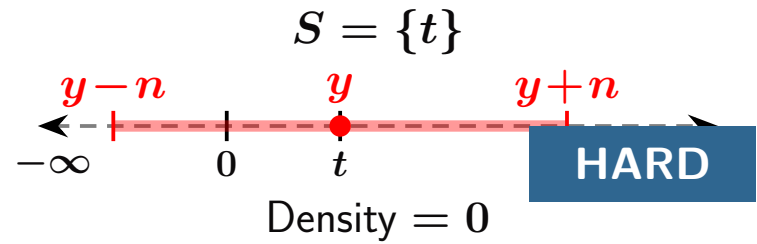
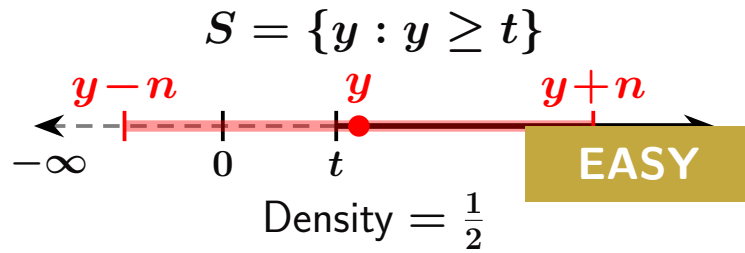
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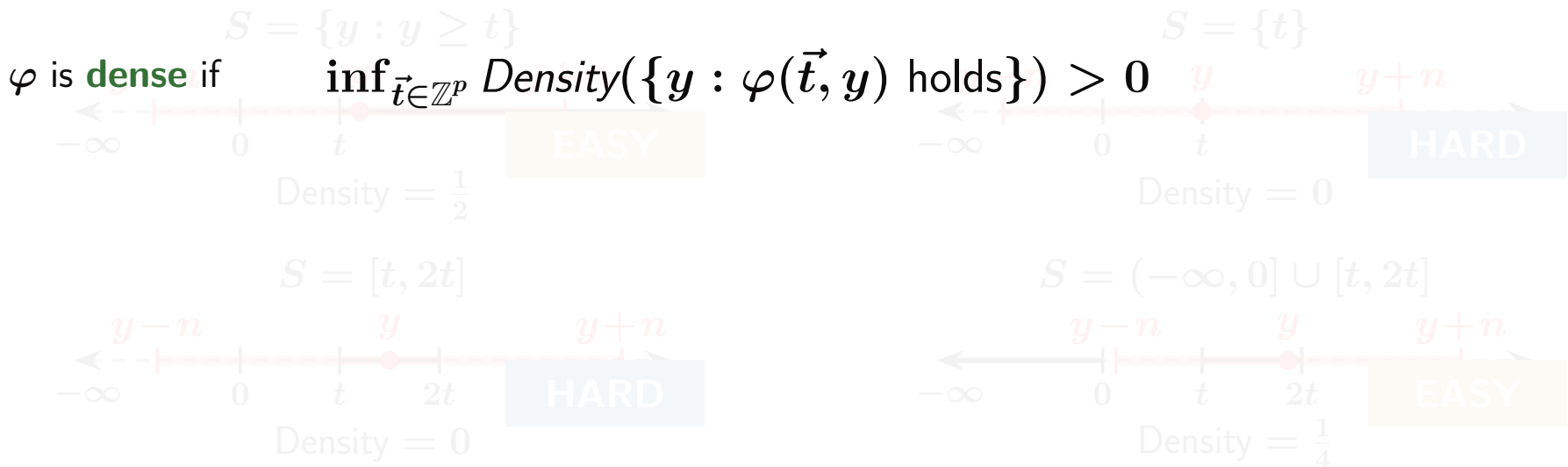
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φ has **zero density**



Reduction to subset sum



φ -REACHABILITY is NP-hard

φ has **positive density** \implies φ -REACHABILITY is in AC^1

Preliminaries: Remove modulo constraints from φ .
Make the integer one-counter automaton acyclic.

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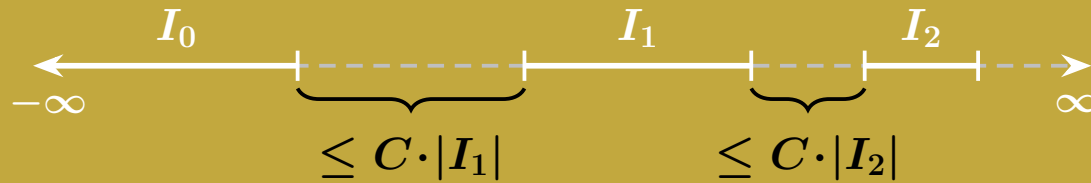


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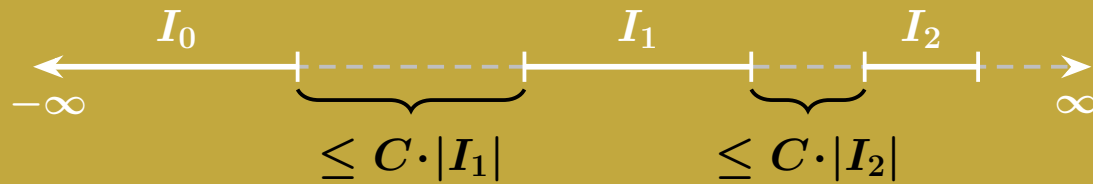
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Design an AC^1 algorithm for reachability to *building blocks*.

View reachability in integer OCA as a semiring problem.

Use the Laurent polynomials with Boolean coefficients $\mathbb{B}[X, X^{-1}]$.

Relies on the technique of repeated squaring.

φ has **zero density**

Reduction to subset sum

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Further Results

Let φ be a Presburger formula with $p + 1$ variables.

Integer semantics: φ -REACHABILITY is ... in AC^1 if φ is **dense**,
NP-hard otherwise.

Natural semantics: φ -REACHABILITY is ... in AC^1 if φ is **dense⁺**,
NP-hard otherwise. ⁺ for a subtly modified definition of density

VASS semantics: φ -REACHABILITY is ... in AC^1 if φ is **uniformly quasi-upward closed**,
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Thank You!



Presented by Henry Sinclair-Banks, University of Warsaw, Poland 

Automata in the Wild 2025, University of Warwick, UK 

Presentation made with
BeamerikZ