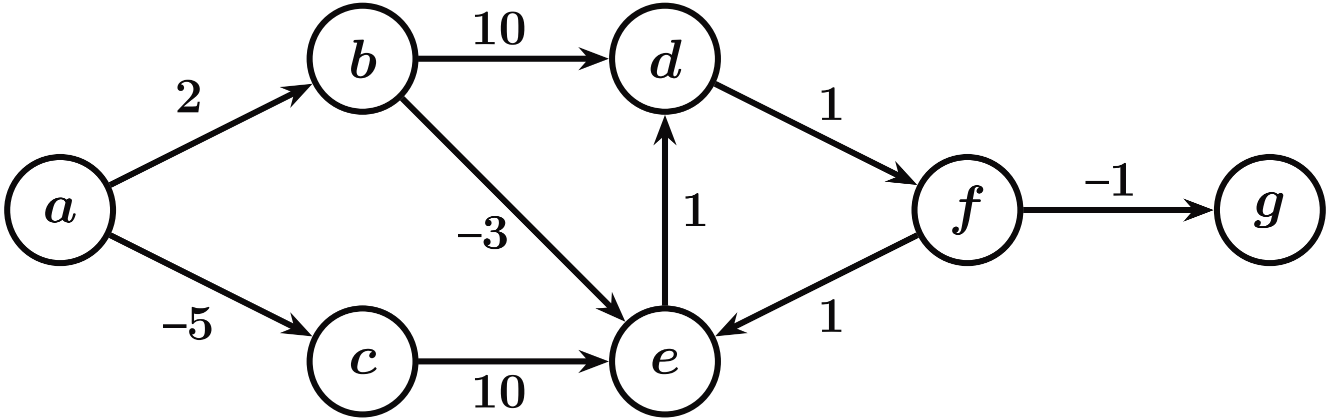
Which Semilinear Target Sets Make Reachability in 1-VASS Easy?

Henry Sinclair-Banks

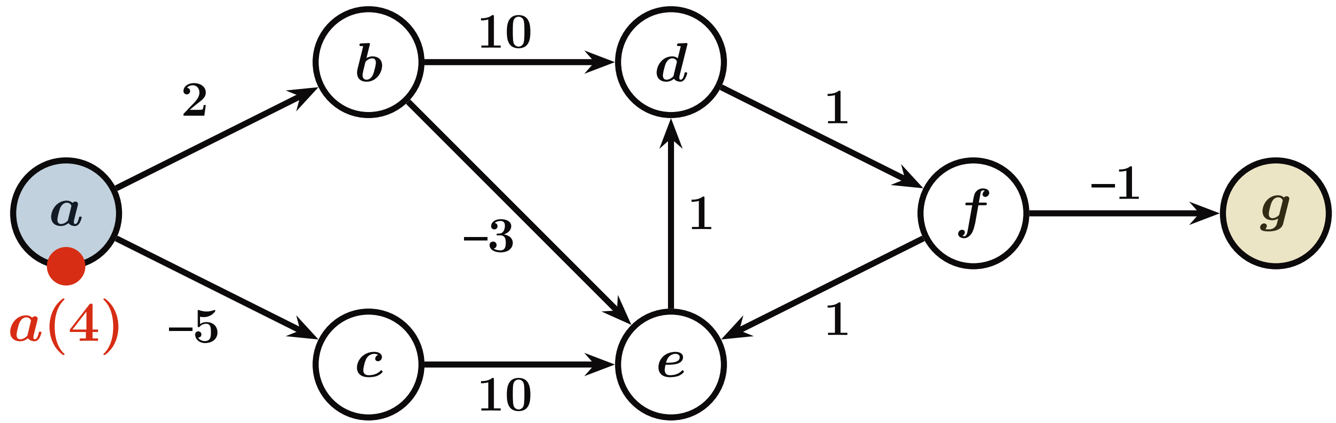
Based on work with Yousef Shakiba and Georg Zetzsche that was accepted to LICS'25.

Highlights'25: Vector Addition Systems Günter-Hotz Hörsaal, Saarland University, Saarbrücken, Germany 3rd September 2025

1-VASS

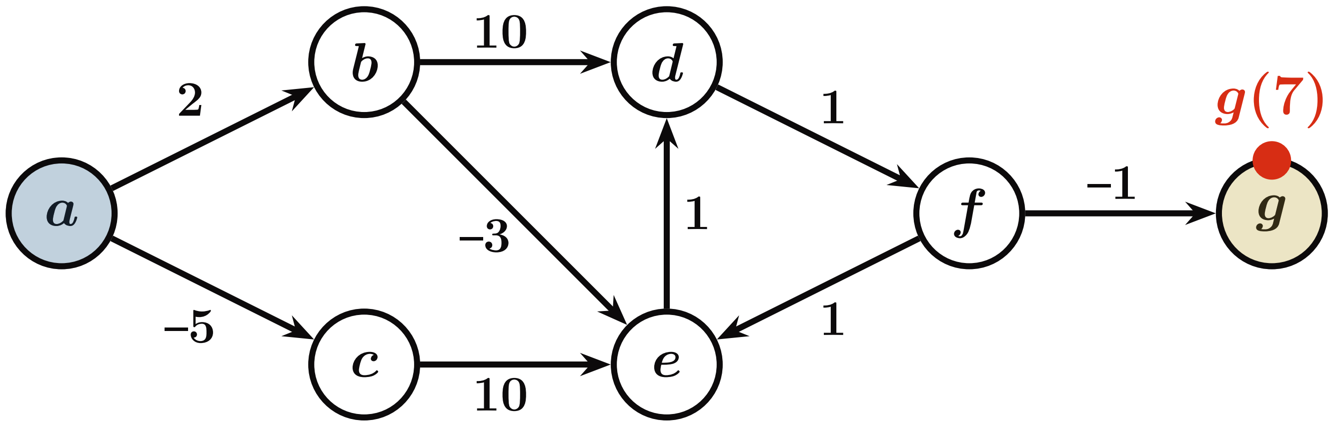


Reachability Problems in 1-VASS



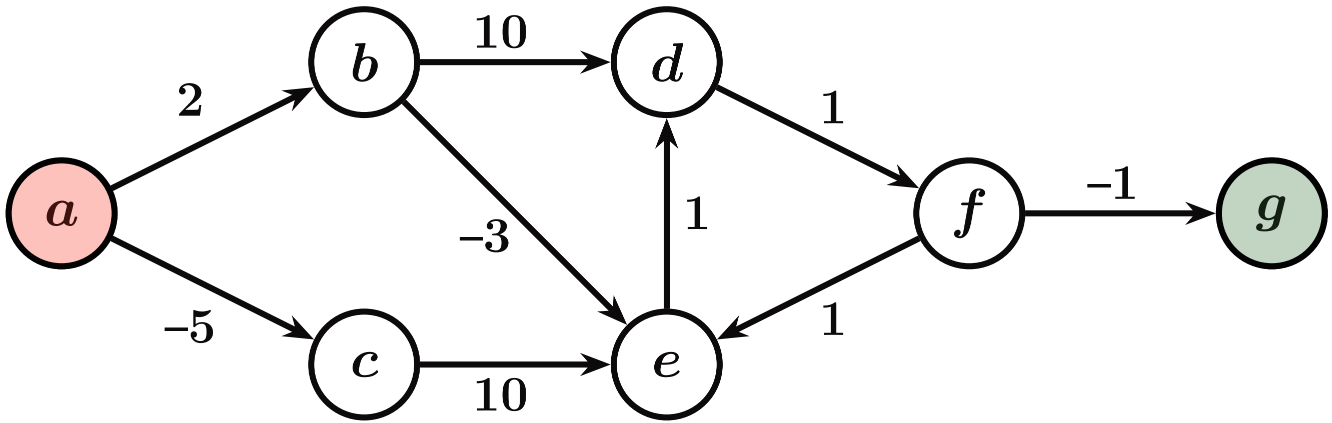
From a with counter value 4, can you reach g with counter value 7?

Reachability Problems in 1-VASS



From a with counter value 4, can you reach g with counter value 7? **Yes!**

Reachability Problems in 1-VASS



From a with counter value 4, can you reach g with counter value 7?

From a with counter value 4, can you reach g with counter value ***at least*** 14? **Yes!**

Motivation

Reachability: In a given 1-VASS, can $\boldsymbol{p}(\boldsymbol{x})$ reach $\boldsymbol{q}(\boldsymbol{y})$?

Theorem. Reachability in (binary encoded) 1-VASS is NP-hard.

Coverability: In a given 1-VASS, can $\boldsymbol{p}(\boldsymbol{x})$ reach $\boldsymbol{q}(\boldsymbol{y}')$ for some $\boldsymbol{y}' \geq \boldsymbol{y}$?

Theorem. Coverability in (binary encoded) 1-VASS is $\mathsf{NC}^2 \subseteq \mathsf{P}$.

Motivation

Reachability: In a given 1-VASS, can $\boldsymbol{p(x)}$ reach $\boldsymbol{q(y)}$?

Theorem. Reachability in (binary encoded) 1-VASS is **HARD**

Coverability: In a given 1-VASS, can $\boldsymbol{p(x)}$ reach $\boldsymbol{q(y')}$ for some $\boldsymbol{y'} \geq \boldsymbol{y}$?

Theorem. Coverability in (binary encoded) 1-VASS is **EASY**

Motivation

Reachability: In a given 1-VASS, can $p(x)$ reach $q(y)$?

Theorem. Reachability in (binary encoded) 1-VASS is **HARD**

Coverability: In a given 1-VASS, can $p(x)$ reach $q(y')$ for some $y' \geq y$?

Theorem. Coverability in (binary encoded) 1-VASS is **EASY**

What makes reachability hard and coverability easy?

Generalising Reachability and Coverability

$\textsc{Reach}(\boldsymbol{S})$

Fixed: A parameterised semilinear set $\boldsymbol{S} \subseteq \mathbb{Z}^{\boldsymbol{p}} \times \mathbb{N}$.

Input: A 1-VASS $\boldsymbol{\mathcal{V}}$, an initial configuration $\boldsymbol{p(x)}$, and parameter values $\vec{\boldsymbol{t}} \in \mathbb{Z}^{\boldsymbol{p}}$.

Question: Does there exist a reachable configuation $\boldsymbol{q(y)}$ such that $(\vec{\boldsymbol{t}}, \boldsymbol{y}) \in \boldsymbol{S}$?

Generalising Reachability and Coverability

$\textsc{Reach}(\boldsymbol{S})$

Fixed: A parameterised semilinear set $\boldsymbol{S} \subseteq \mathbb{Z}^{\boldsymbol{p}} \times \mathbb{N}$.

Input: A 1-VASS $\boldsymbol{\mathcal{V}}$, an initial configuration $\boldsymbol{p}(\boldsymbol{x})$, and parameter values $\vec{\boldsymbol{t}} \in \mathbb{Z}^{\boldsymbol{p}}$.

Question: Does there exist a reachable configuation $\boldsymbol{q}(\boldsymbol{y})$ such that $(\vec{\boldsymbol{t}}, \boldsymbol{y}) \in \boldsymbol{S}$?

Examples

Reachability: $\boldsymbol{S} = \{(\boldsymbol{t}, \boldsymbol{y}) : \boldsymbol{y} = \boldsymbol{t}\}$

Coverability: $\boldsymbol{S} = \{(\boldsymbol{t}, \boldsymbol{y}) : \boldsymbol{y} \geq \boldsymbol{t}\}$

Cover and avoid: $\boldsymbol{S} = \{(\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{y}) : \boldsymbol{y} \geq \boldsymbol{t}_1 \wedge \boldsymbol{y} \neq \boldsymbol{t}_2\}$

Reach an interval: $\boldsymbol{S} = \{(\boldsymbol{t}, \boldsymbol{y}) : \boldsymbol{t} \leq \boldsymbol{y} \leq \boldsymbol{2t}\}$

Generalising Reachability and Coverability

$\textsc{Reach}(S)$

Fixed: A parameterised semilinear set $S \subseteq \mathbb{Z}^p \times \mathbb{N}$.

Input: A 1-VASS $\mathcal{V}$, an initial configuration $p(x)$, and parameter values $\vec{t} \in \mathbb{Z}^p$.

Question: Does there exist a reachable configuation $q(y)$ such that $(\vec{t}, y) \in S$?

Examples

Reachability: $S = \{(t, y) : y = t\}$	HARD
Coverability: $S = \{(t, y) : y \geq t\}$	EASY
Cover and avoid: $S = \{(t_1, t_2, y) : y \geq t_1 \wedge y \neq t_2\}$	EASY
Reach an interval: $S = \{(t, y) : t \leq y \leq 2t\}$	HARD

Main Contribution

Theorem. Let $\boldsymbol{S} \subseteq \mathbb{Z}^{\boldsymbol{p}} \times \mathbb{N}$ be a semilinear set.

(1) If $\boldsymbol{S}$ is **uniformly quasi-upwards closed**, then $\textsc{Reach}(\boldsymbol{S})$ is **EASY**

(2) Otherwise, $\textsc{Reach}(\boldsymbol{S})$ is **HARD**

Main Contribution

Theorem. Let $S \subseteq \mathbb{Z}^p \times \mathbb{N}$ be a semilinear set.

(1) If S is **uniformly quasi-upwards closed**, then $\textsc{Reach}(S)$ is **EASY**

(2) Otherwise, $\textsc{Reach}(S)$ is **HARD**

Main Contribution

Theorem. Let $S \subseteq \mathbb{Z}^p \times \mathbb{N}$ be a semilinear set.

(1) If S is **uniformly quasi-upwards closed**, then $\textsc{Reach}(S)$ is **EASY**

(2) Otherwise, $\textsc{Reach}(S)$ is **HARD**

Upwards closed: $\{10, 11, 12, 13, \ldots\}$.

δ-upwards closed: $\{10, 15, 17, 20, 22, 25, 27, \ldots\}$ is 5-upwards closed.

(δ, M)-upwards closed: $\{10, 17, 20, 22, 27, 30, 32, 35, \ldots\}$ is $(5, 2)$-upwards closed.

Uniformly quasi-upwards closed

There exists $\delta, M \in \mathbb{N}$ such that, for all $\vec{t} \in \mathbb{Z}^p$, $\{y : (\vec{t}, y) \in S\}$ is (δ, M)-upwards closed.

Main Contribution

Theorem. Let $S \subseteq \mathbb{Z}^p \times \mathbb{N}$ be a semilinear set.

(1) If S is **uniformly quasi-upwards closed**, then $\textsc{Reach}(S)$ is **EASY**

(2) Otherwise, $\textsc{Reach}(S)$ is **HARD**

Upwards closed: $\{10, 11, 12, 13, \ldots\}$.

δ-upwards closed: $\{10, 15, 17, 20, 22, 25, 27, \ldots\}$ is 5-upwards closed.

(δ, M)-upwards closed: $\{10, 17, 20, 22, 27, 30, 32, 35, \ldots\}$ is $(5, 2)$-upwards closed.

Uniformly quasi-upwards closed

There exists $\delta, M \in \mathbb{N}$ such that, for all $\vec{t} \in \mathbb{Z}^p$, $\{y : (\vec{t}, y) \in S\}$ is (δ, M)-upwards closed.

Main Contribution

Theorem. Let $S \subseteq \mathbb{Z}^p \times \mathbb{N}$ be a semilinear set.

(1) If S is **uniformly quasi-upwards closed**, then $\textsc{Reach}(S)$ is in AC^1.

(2) Otherwise, $\textsc{Reach}(S)$ is NP-hard.

Upwards closed: $\{10, 11, 12, 13, \ldots\}$.

δ-upwards closed: $\{10, 15, 17, 20, 22, 25, 27, \ldots\}$ is 5-upwards closed.

(δ, M)-upwards closed: $\{10, 17, 20, 22, 27, 30, 32, 35, \ldots\}$ is $(5, 2)$-upwards closed.

Uniformly quasi-upwards closed

There exists $\delta, M \in \mathbb{N}$ such that, for all $\vec{t} \in \mathbb{Z}^p$, $\{y : (\vec{t}, y) \in S\}$ is (δ, M)-upwards closed.

If S is **uniformly quasi-upwards closed**, then $\textsc{Reach}(S)$ is in $\mathbf{AC}^1$.

If S is **uniformly quasi-upwards closed**, then $\textsc{Reach}(S)$ is in **$\mathbf{AC}^1$**.

If S is **uniformly quasi-upwards closed**, then $\textsc{Reach}(S)$ is in **AC^1**.

Theorem. Coverability in (binary encoded) 1-VASS is in NC^2.

[Almagor, Cohen, Pérez, Shirmohammadi, and Worrell 2020]

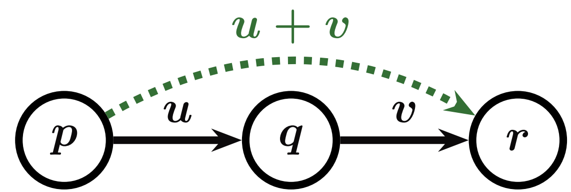
An Efficient Algorithm for Coverability in 1-VASS

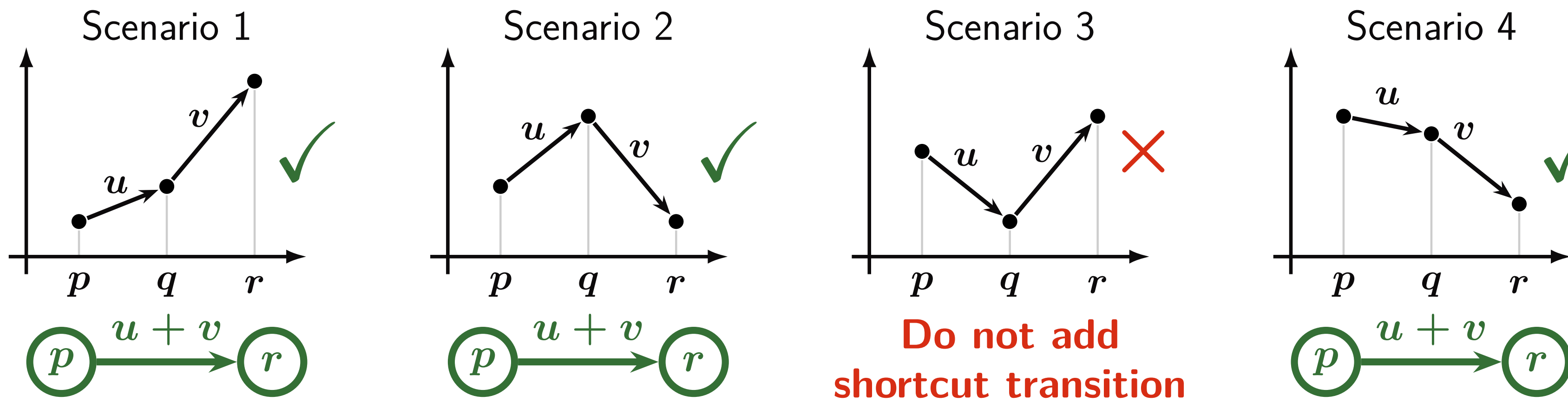
`STEP 0:` Let $\mathcal{V}_0$ be the given 1-VASS.

An Efficient Algorithm for Coverability in 1-VASS

`STEP 0:` Let $\mathcal{V}_0$ be the given 1-VASS.

`STEP 1:` Create $\mathcal{V}_1$ from $\mathcal{V}_0$ by adding "shortcut transitions":

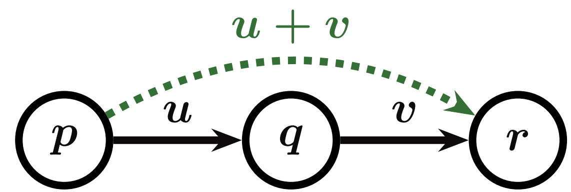


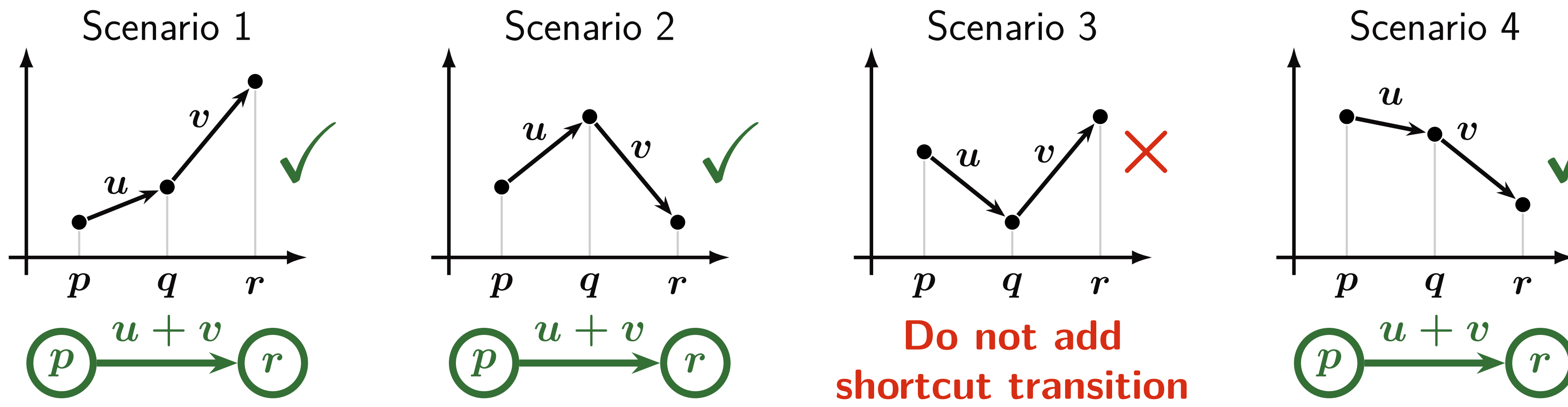


An Efficient Algorithm for Coverability in 1-VASS

`STEP 0:` Let $\mathcal{V}_0$ be the given 1-VASS.

`STEP 1:` Create $\mathcal{V}_1$ from $\mathcal{V}_0$ by adding "shortcut transitions":

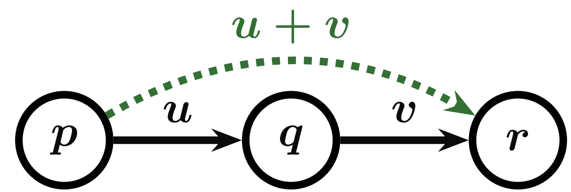




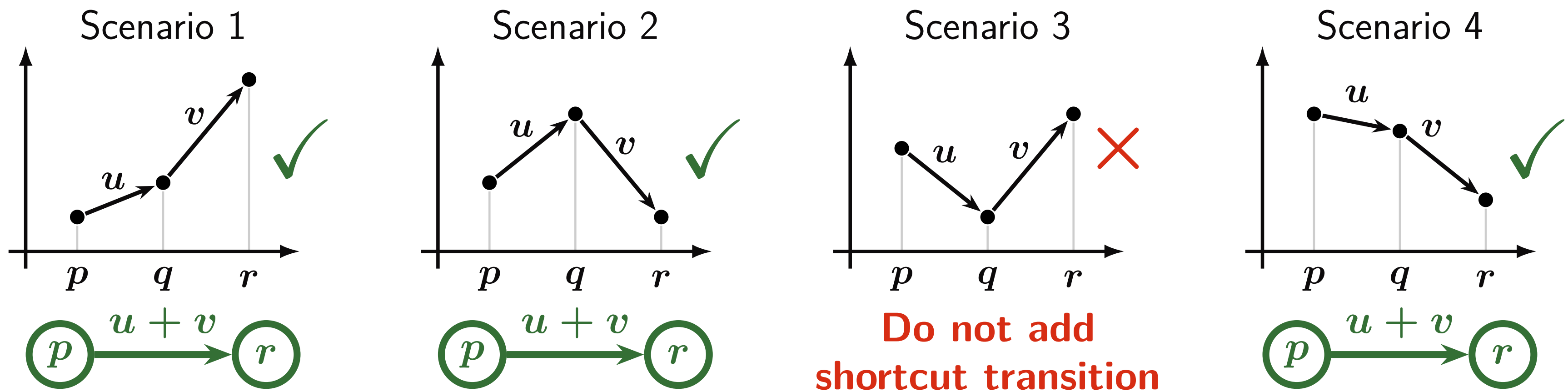
`STEP i:` Repeat $k = 2\lceil \log n \rceil$ many times.

An Efficient Algorithm for Coverability in 1-VASS

`STEP 0:` Let $\mathcal{V}_0$ be the given 1-VASS.



`STEP 1:` Create $\mathcal{V}_1$ from $\mathcal{V}_0$ by adding "shortcut transitions":



`STEP i:` Repeat $k = 2\lceil \log n \rceil$ many times.

Claim. There is a covering run in $\mathcal{V}_0$ if and only if there is a covering run of **length ≤ 2** in $\mathcal{V}_k$.

Which Semilinear Target Sets Make Reachability in 1-VASS Easy?

$\textsc{Reach}(\boldsymbol{S})$

Fixed: A parameterised semilinear set $\boldsymbol{S} \subseteq \mathbb{Z}^{\boldsymbol{p}} \times \mathbb{N}$.

Input: A 1-VASS $\boldsymbol{\mathcal{V}}$, an initial configuration $\boldsymbol{p}(\boldsymbol{x})$, and parameter values $\vec{\boldsymbol{t}} \in \mathbb{Z}^{\boldsymbol{p}}$.

Question: Does there exist a reachable configuation $\boldsymbol{q}(\boldsymbol{y})$ such that $(\vec{\boldsymbol{t}}, \boldsymbol{y}) \in \boldsymbol{S}$?

Theorem. Let $\boldsymbol{S} \subseteq \mathbb{Z}^{\boldsymbol{p}} \times \mathbb{N}$ be a semilinear set.

(1) If $\boldsymbol{S}$ is **uniformly quasi-upwards closed**, then $\textsc{Reach}(\boldsymbol{S})$ is in AC^1.

(2) Otherwise, $\textsc{Reach}(\boldsymbol{S})$ is NP-hard.

Thank You!

Presented by Henry Sinclair-Banks, University of Warsaw, Poland

Highlights'25 in Saarland University, Saarbrücken, Germany

Presentation made with BeamerikZ