Infinite Automata 2025/26

Final Examination

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Format

This document contains all questions that can be asked during the examination; they are partitioned into list A and list B. In the exam, you will first be invited to present a solution to one question from list A; **you can choose whichever question you like to answer from list A**. After, for the second half of the exam, you will be asked to present solutions of (up to) four questions from list B; **you will not be given a choice over which list B questions are asked**. Question lists A and B can be found on the second and third pages of this document, respectively.

Timing and Scheduling

Examinations will take place within an hour. Here is an example (approximate) timeline for an examination that takes place between 10:00 and 11:00.

- Please arrive by 10:00.
- The exam will begin at 10:05.
- Between 10:05 and 10:25, for up to twenty minutes, you are invited to present a solution to a question from list A.
- Between 10:25 and 10:50, for up to twenty-five minutes, you are then asked to present solutions to questions from list B.
- Between 10:50 and 10:55, there will be an opportunity to quickly discuss the questions should you wish and you can receive immediate feedback (see the Grading subsection).
- The exam will conclude before 11:00.

The preliminary examination dates are 5th and 6th February 2026. Email Henry to schedule your examination, please include "Infinite Automata" in the subject line.

Grading

At the end of an examination, if you wish, you can immediately receive feedback that is based on your examination score. The examination score is based on: how well the question from list A is answered, how many questions from list B are answered without hints, and how many questions from list B are answered with hints. You can score up to five points when presenting a solution to a question from list A. Higher scores are given to students who can: demonstrate their understanding of a correct solution; accurately provide a convincing solution in a timely manor; and explain additional details on request. You can score up to two points for answering a question from list B. Two points are awarded when the student is able to convincingly answer a question without (or only with minimal) prompts. One point is awarded when the student is able to answer the question but either makes a moderate mistake, does not demonstrate their understanding very clearly, or requires a hint before correctly answering the question.

Examinations will be graded *after* all examinations have taken place. The final grade for this course will take your examination score and any additional scores from starred exercises into account. You will be notified of your final grade by email shortly after the deadline for Series 2 of the starred exercises (or after the last examination, whichever occurs last).

15th January 2026

Question List A

You will be invited to present a solution to a question of your choice from this list.

Question A.1. *Reachability in 1-VASS.*
Explain what the size of a binary-encoded 1-VASS is, and prove that reachability in binary-encoded 1-VASS is NP-complete.

Remark. See Lecture Notes 3.

Question A.2. *Karp-Miller trees.*
Explain Dickson's lemma, define Karp-Miller trees, prove that Karp-Miller trees are finite, and show how they can be used to prove that unboundedness in VASS is decidable.

Remark. See Lecture Notes 4 and 5.

Question A.3. *Parikh's theorem.*
Define the Parikh function, state Parikh's theorem, and prove Parikh's theorem. You may use Lemma 6.6 without a proof (but if you wish to use it, you should state it).

Remark. See Lecture Notes 6 and 7.

Question A.4. *Pottier's lemma.*
Explain what homogeneous systems of integer linear equations are, state Pottier's lemma, and prove Pottier's lemma.

Remark. See Lecture Notes 7.

Question A.5. *Rackoff's bound for coverability.*
Define the coverability problem for VASS; prove that if coverability holds, then there exists a doubly exponential length run witnessing coverability; and show that coverability in VASS can be decided in exponential space.

Remark. See Lecture Notes 9 and 10.

Question List B

You will be asked to present solutions of up to four questions from this list (not of your choice).

Question B.1. Given multisets A and B each comprising of at least $2n$ positive integers not greater than n, show that there exists non-empty subsets $A' \subseteq A$ and $B' \subseteq B$ such that the sum of elements in A' equals to the sum of elements in B' (Exercise 1.2).

Question B.2. Prove that reachability in two-counter machines is undecidable (Exercise 3.6).

Question B.3. State the three conditions that makes $(X, \preceq)$ a well-quasi order and prove that they are equivalent (Definition 4.1 and Lemma 4.3).

Question B.4. State and prove infinite Ramsey's theorem (Theorem 4.2).

Question B.5. State and prove König's lemma (Lemma 4.4, Exercise 4.2).

Question B.6. State and prove Dickson's lemma (Lemma 4.5, Exercise 4.4).

Question B.7. Provide an example of a d-VASS $V = (Q, T)$, and a configuration $(p, \mathbf{u})$ such that the number of configurations that are reachable from $(p, \mathbf{u})$ is finite but exceeds $F_{d-1}(c \cdot n)$, where n is the size of V and $(p, \mathbf{u})$ encoded in unary, c is some constant that is independent of V and $(p, \mathbf{u})$, and F_{d-1} is the $(d-1)$-st fast growing function (Exercise 5.9).

Question B.8. Show that reachability in d-VASS can be reduced (in logarithmic space) to reachability in $(d+3)$-VAS (Exercise 6.2).

Question B.9. Prove that $\{(n, m) : n \in \mathbb{N} \text{ and } m \leq 2^n\}$ is not semilinear (Exercise 6.3.4).

Question B.10. Using Parikh's theorem, prove that languages recognised by pushdown automata over a unary alphabet ($\Sigma = \{a\}$) are regular (Corollary 6.7).

Question B.11. State Pottier's lemma (Lemma 7.2) and use it to prove that reachability in integer VASS (where the counters are allowed to drop below zero) is in NP.

Question B.12. Define semilinear sets and show that semilinear sets are closed under intersection (Exercise 7.3).

Questions B.13. Define subset-sum games (Definition 8.6), explain counter-stack automata (Definition 8.3), and define bounded 1-VASS (Definition 8.1); and prove that reachability in bounded 1-VASS is in PSPACE.

Question B.14. Prove that coverability in binary-encoded 1-VASS is decidable in polynomial time (Exercise 9.5).

Question B.15. State the "the controlling counter technique" lemma (see Exercise Sheet 11) and prove that the shortest runs that witness reachability in unary-encoded 3-VASS can have exponential length (Exercise 11.1).

Question B.16. Let $(p, \mathbf{u})$ and $(q, \mathbf{v})$ be two configurations in a VASS V, let $\mathbf{p} \geq (1, 1, \ldots, 1)$. Prove that reachability from $(p, \mathbf{u})$ to $(q, \mathbf{v})$ is decidable, assuming that $(p, \mathbf{u}) \xrightarrow{*} (p, \mathbf{u} + \mathbf{p})$ and $(q, \mathbf{v} + \mathbf{p}) \xrightarrow{*} (q, \mathbf{v})$ (Exercise 13.2).