Infinite Automata 2025/26

Lecture Notes 4

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Definition 4.1. Let $\preceq$ be a relation over an infinite set X that is reflexive and transitive (that is not necessarily antisymmetric). The pair $(X, \preceq)$ is a *well-quasi order (WQO)* if it satisfies any of the following conditions.

(1) There does not exist an infinite antichain or an infinite strictly decreasing sequence in X.

(2) In every infinite sequence $(x_i)_{i=1}^{\infty}$ in X, there exist indices i and j such that $i < j$ and $x_i \preceq x_j$.

(3) In every infinite sequence $(x_i)_{i=1}^{\infty}$ in X, there exists an infinite subsequence $x_{i_1} \preceq x_{i_2} \preceq x_{i_3} \ldots$.

Note that *strictly decreasing* here means: $x \prec y$ if and only if $x \preceq y$ and $y \not\preceq x$.

Theorem 4.2. *Infinite Ramsey's Theorem.* Let $G = (V, E)$ be an infinite complete graph. Fix $k \in \mathbb{N}$. For every k-edge-colouring of G, there exists an infinite monochromatic clique in G.

Proof. Let $c : E \to \{1, \ldots k\}$ be a k-edge-colouring of G. Let $S_1 = \left(v_i^1\right)_{i=1}^{\infty}$ be an infinite sequence of distinct nodes of G. Since G is complete, we know that for all $j > 1$, $\{v_1^1, v_j^1\} \in E$ (which means v_1^1 has infinitely many neighbours in S_1). By pigeonhole principle, there exists a colour $c_1 \in \{1, \ldots, k\}$ such that infinitely many such edges are c_1-coloured.

Now, let $S_2 = \left(v_i^2\right)_{i=1}^{\infty}$ be the infinite subsequence of S_1 such that for all $j \in \mathbb{N}$, the edge between v_1^1 and v_j^2 is c_1-coloured. Like before, since G is complete, we know that for all $j > 1$, $\{v_1^2, v_j^2\} \in E$ (which means v_1^2 has infinitely many neighbours in S_2). By pigeonhole principle, there exists a colour $c_2 \in \{1, \ldots, k\}$ such that infinitely many such edges are c_2-coloured.

This argument repeats. Now, we have an infinite sequence of distinguished nodes $v_1^1, v_1^2, v_1^3, \ldots$. Each of these nodes v_1^i has the colour c_i associated to it. Since there are k colours, there must exist a colour $c_\infty \in \{1, \ldots, k\}$ that occurs infinitely often in $c_1, c_2, c_3, \ldots$. Take the infinite subsequence $(c_{i_j})_{j=1}^{\infty}$ such that $c_{i_1} = c_{i_2} = c_{i_3} = \ldots = c_\infty$.

We now claim that the subsequence $C = (v_1^{i_j})_{j=1}^{\infty}$ forms a c_∞-coloured (monochromatic) infinite clique. To see why this is the case, first consider S_{i_1}. All vertices $v_1^{i_2}, v_1^{i_3}, \ldots$ belong to S_{i_1} and so we know that $\{v_1^{i_1}, v_1^{i_j}\} \in E$ such that $j > 1$ have the colour $c_1 = c_\infty$. More generally, for every $j \in \mathbb{N}$, the sequence S_{i_j} contains all vertices $v_1^{i_{j+1}}, v_1^{i_{j+2}}, \ldots$ and so we know that all edges $\{v_1^{i_j}, v_1^{i_{j'}}\} \in E$ such that $j' > j$ have the colour $c_{i_j} = c_\infty$. □

Lemma 4.3. The three conditions (1), (2), and (3) in Definition 4.1 are equivalent.

Proof. To prove these statements are equivalent, we will prove (1) $\implies$ (3), (3) $\implies$ (2), and (2) $\implies$ (1).

The easiest case is (3) $\implies$ (2). If every infinite sequence $(x_i)_{i=1}^{\infty}$ in X has an infinite subsequence $x_{i_1} \preceq x_{i_2} \preceq x_{i_3} \preceq \ldots$, then trivially there exists $i = i_1$ and $j = i_2$ such that $i < j$ and $x_i \preceq x_j$.

The next easiest case is (2) $\implies$ (1). This is equivalent to showing that $\neg(1) \implies \neg(2)$. So $\neg(1)$ means that there either exists an infinite antichain or an infinite strictly decreasing sequence in X. Regardless, suppose $(x_i)_{i=1}^{\infty}$ is the infinite antichain or infinite strictly decreasing sequence in X. It is immediate that there does not exist an $i, j \in \mathbb{N}$ such that $i < j$ and $x_i \preceq x_j$ which means that condition (2) is false.

Lastly, we will use Infinite Ramsey's Theorem to prove that (1) $\implies$ (3). Take any infinite sequence $S = (x_i)_{i=1}^{\infty}$ in X. Create an infinite complete graph $G = (S, E)$ and a 3-edge-colouring for G as follows. Let $\{\text{red}, \text{green}, \text{blue}\}$ be the set of colours. Take a pair of indices i and j such that $i < j$,

- if $x_i \preceq x_j$, then colour $\{x_i, x_j\}$ green;
- else if $x_j \prec x_i$, then colour $\{x_i, x_j\}$ red; and
- otherwise if x_i and x_j do not compare, then colour $\{x_i, x_j\}$ blue.

Now, assuming (1), we know that there cannot be an infinite red clique because there does not exist an infinite strictly decreasing sequence in X and there cannot be an infinite blue clique because there does not exist an infinite antichain. Using Theorem 4.2, we know that there must exist a monochromatic clique in G with this 3-edge-colouring. We conclude that such a monochromatic clique must be green and so there is an infinite subsequence in X: $x_{i_1} \preceq x_{i_2} \preceq x_{i_3} \preceq \ldots$. □

We shall now examine a pair of applications of well-quasi orders. We will prove that it is decidable whether a given configuration can reach an infinite number of configurations in a given VASS. We will also prove that coverability in VASS is decidable. First, we shall state two preliminary lemmas. Recall Exercise 4.2 and Exercise 4.4.

Lemma 4.4. *König's Lemma, specialised to trees.* Let G be a tree that has finite branching (is locally finite) that does not contain an infinite path, then G itself is finite.

Lemma 4.5. *Dickson's Lemma.* Let $X \subseteq \mathbb{N}^d$ be nonempty. The pair $(X, \leq)$ where $\leq$ is point-wise inequality ordering is a WQO. It is also true that the set of minimal elements of X (the set of elements $\mathbf{x} \in X$ such that there does not exist $\mathbf{y} \in X$ such that $\mathbf{y} \leq \mathbf{x}$) is finite.

Theorem 4.6. Let $V = (Q, T)$ be a VASS and let $(p, \mathbf{u})$ be a configuration. It is decidable whether the set of configurations $\{(q, \mathbf{v}) : (p, \mathbf{u}) \xrightarrow{*}_V (q, \mathbf{v})\}$ is finite.

Proof. Consider a tree of configurations that is rooted at $(p, \mathbf{u})$. The children of a node $(s, \mathbf{x})$ are configurations $(t, \mathbf{y})$ such that $(s, \mathbf{x}) \to_V (t, \mathbf{y})$. This tree is finitely branching because, for any given node, it has at most $|T|$ many children.

For a moment, consider the infinite tree of reachable configuration from $(p, \mathbf{u})$; there may be nodes at different levels of the tree that correspond to the same configuration. Consider some infinite path through this tree $(p, \mathbf{u}) = (q_0, \mathbf{v}_0), (q_1, \mathbf{v}_1), (q_2, \mathbf{v}_2), \ldots$. Since there are only $|Q|$ many different nodes, there exists a subsequence of these configurations that share the same state $s \in Q$ and since the original sequence of configurations was a run, the following configurations can be reached in sequence: $(s, \mathbf{v}_{i_1}), (s, \mathbf{v}_{i_2}), (s, \mathbf{v}_{i_3}), \ldots$. Since $(\mathbb{N}^d, \leq)$ is a WQO, we know that there must exist indices j, j' such that $j < j'$ such that $\mathbf{v}_{i_j} \leq \mathbf{v}_{i_{j'}}$.

Now, consider the DFS-style algorithm that exhaustively creates this tree of reachable configurations. By the above argument, we know that every path down the tree will eventually witness a pair of configurations $(s, \mathbf{x})$ and $(s, \mathbf{y})$ such that $(p, \mathbf{u}) \xrightarrow{*}_v (s, \mathbf{x}) \xrightarrow{*}_V (s, \mathbf{y})$ and $\mathbf{x} \leq \mathbf{y}$. There are two cases. First, if $\mathbf{y} = \mathbf{x}$, then the algorithm stops exploring beyond this configuration $(s, \mathbf{y})$. This is because repeatedly exploring beyond this point will not reveal any distinct reachable configurations. Second, if $\mathbf{y} > \mathbf{x}$, then the algorithm immediately terminates and outputs "no" (the set of reachable configurations from $(p, \mathbf{u})$ if infinite). The algorithm continues to explore and generate the tree; this can only continue for a finite number of steps due to Lemma 4.4 (the tree is finitely branching and does not contain infinite paths). If after all paths have been exhausted and in every case $\mathbf{x} = \mathbf{y}$ was observed, then the algorithm outputs "yes" (the set of reachable configuration from $(p, \mathbf{u})$ is finite). □

Definition 4.7. Coverability in VASS (problem).

Input. A VASS V, an initial configuration $(p, \mathbf{u})$, and a target configuration $(q, \mathbf{v})$.

Question. Does there exist a run $(p, \mathbf{u}) \xrightarrow{*}_V (q, \mathbf{v}')$ such that $\mathbf{v}' \geq \mathbf{v}$?

For the following definitions and claims, we will fix our attention to an instance of coverability which consists of a d-VASS $V = (Q, T)$, an initial configuration $(p, \mathbf{u})$, and a target configuration $(q, \mathbf{v})$.

Definition 4.8. A *coverability separator*, for $(V, (p, \mathbf{u}), (q, \mathbf{v}))$, is a set of configurations $S \subseteq Q \times \mathbb{N}^d$ that satisfies the following four properties:

(1) S is upward closed ($S = S\uparrow$);

(2) $(p, \mathbf{u}) \notin S$;

(3) $(q, \mathbf{v}) \in S$; and

(4) for all pairs of configurations c, c' such that $c \to_V c'$, if $c' \in S$, then $c \in S$.

In property (1), we use the notiation $X\uparrow$ to denote the upward closure of a set X; precisely

$$X\uparrow := \{y : \text{there exists } x \in X \text{ such that } y \geq x\}.$$

Claim 4.9. If there does not exist a run from $(p, \mathbf{u})$ to a configuration $(q, \mathbf{v}')$ in V such that $\mathbf{v}' \geq \mathbf{v}$, then there exists a coverability separator S for $(V, (p, \mathbf{u}), (q, \mathbf{v}))$.

Proof. Consider the set of configurations which can cover $(q, \mathbf{v})$:

$$C = \{c \in Q \times \mathbb{N}^d : c \xrightarrow{*}_V (q, \mathbf{v}') \text{ for some } \mathbf{v}' \geq \mathbf{v}\}.$$

We will now argue that C is a coverability separator for $(V, (p, \mathbf{u}), (q, \mathbf{v}))$. Firstly, to see why C is upward closed, consider any configuration $(s, \mathbf{x}) \in C$. There exists $\mathbf{v}' \geq \mathbf{v}$ such that there is a run $(s, \mathbf{x}) \xrightarrow{\pi}_V (q, \mathbf{v}')$. Now consider any $\mathbf{y} \geq \mathbf{x}$ and let $\Delta = \mathbf{y} - \mathbf{x}$; clearly $\Delta \geq \mathbf{0}$. By monotonicity, we know that $(s, \mathbf{y}) \xrightarrow{\pi}_V (q, \mathbf{v}' + \Delta)$ and since $\mathbf{v}' + \Delta \geq \mathbf{v}' \geq \mathbf{v}$, we conclude that $(s, \mathbf{y}) \in C$ as well. Thus C satisfies property (1).

It is immediate that $(p, \mathbf{u}) \notin C$ by assumption that this instance of coverability is negative. This means that property (2) holds. Moreover, property (3) trivially holds because $(q, \mathbf{v}) \in C$.

Lastly, for property (4), if there are a pair of configurations $c, c' \in Q \times \mathbb{N}^d$ such that $c \to_V c'$, then if $c' \in C$, then it is true that $c' \xrightarrow{*}_V (q, \mathbf{v}')$ for some $\mathbf{v}' \geq \mathbf{v}$. Hence $c \to_V c' \xrightarrow{*}_V (q, \mathbf{v}')$ implies that $c \in C$. □

Claim 4.10. If X, Y be two upward closed sets, then $X \cap Y$ is upward closed.

Proof. Let $x \in X \cap Y$ be an arbitrary point. Consider any point y such that $y \geq x$. Since X is upward closed, $x \in X$ implies that $y \in X$; and since Y is upward closed, $x \in Y$ implies that $y \in Y$. Hence $y \in X \cap Y$ and so we conclude that $X \cap Y$ is upward closed. □

Theorem 4.11. Coverability in VASS is decidable.

Proof. We will design two semi-decision procedures for deciding whether there is a run $(p, \mathbf{u}) \xrightarrow{*}_V (q, \mathbf{v}')$ for some $\mathbf{v}' \geq \mathbf{v}$ or not. One procedure will terminate after finitely many steps if coverability holds and the other procedure will terminate after finitely many steps if coverability does not hold. We obtain an algorithm for coverability in VASS by running these two semi-decision procedures "in parallel".

The first procedure simply builds the tree of reachable configurations from $(p, \mathbf{u})$ using a breadth-first approach. If at any point, a configuration $(q, \mathbf{v}')$ is found such that $\mathbf{v}' \geq \mathbf{v}$ then this procedure terminates and outputs "yes" (coverability holds). It is immediate that if coverability holds, then this first procedure terminates after finitely many steps.

The second procedure will attempt to find a coverability separator. By Claim 4.9, we know that if coverability does not hold, then there exists a coverability separator $C \subseteq Q \times \mathbb{N}^d$. Since C is upward closed, we know that it can be represented as the upward closure of its minimal elements M; Lemma 4.5 tells us that there are only finitely many minimal elements for the upward closed set C. Since finite sets configurations in $Q \times \mathbb{N}^d$ are countably infinite, there exists an enumeration procedure which can iterate through every possible set of minimal elements.

It remains to argue that given a finite set $M \subset Q \times \mathbb{N}^d$, it is decidable whether $M\uparrow$ is a coverability separator for $(V, (p, \mathbf{u}), (q, \mathbf{v}))$. We need not check property (1). Property (2) is easy to check: it suffices to check that $(p, \mathbf{u}) \not\geq m$, for all $m \in M$. Similarly for property (3), it suffices to check whether there exists $m \in M$ such that $(q, \mathbf{v}) \geq m$. Lastly, we wish to prove property (4): let c, c' be configurations such that $c \rightarrow_V c'$, if $c' \in M\uparrow$, then $c \in M\uparrow$. It would suffice to check this condition only for $c' \in M$, however it might be the case that in order to take a transition t to reach c', one of the counters would have needed to have a negative value. To get around this, we need to check that property (4) holds for configurations c' that belong to $M\uparrow$ that can be reached from valid configurations. Let $t \in T$ be an arbitrary transition. Let A_t be the set of configurations from which t can be taken, $A_t \coloneqq \{(s, \mathbf{x}) \in Q \times \mathbb{N}^d : \mathbf{x} + \text{effect}(t) \geq \mathbf{0}\}$. Note that A_t is an upward closed set. Now consider $A_t \cap (M\uparrow)$; by Claim 4.10, we know that this set is upward closed and by Lemma 4.5, we know that there are only finitely many minimal points m' in $A_t \cap (M\uparrow)$. It suffices to only check property (4) for these minimal elements m'. This requires checking whether the configuration $m' - \text{effect}(t)$ belongs to $M\uparrow$ (by checking if there exists $m \in M$ such that $m \leq m'$).

To conclude, the second procedure will enumerate over candidate finite representations (sets of minimal elements) of coverability separators and for every such set of minimal elements, it checks whether indeed their upward closure is a coverability separator for $(V, (p, \mathbf{u}), (q, \mathbf{v}))$. If it finds such a separator, then the procedure outputs “no” (coverability does not hold). This procedure will terminate after finitely many steps if coverability indeed does not hold. ☐