Infinite Automata 2025/26

Exercise Sheet 13

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Exercise 13.1. Let $C \in \mathbb{N}$. Create C^n-amplifier that uses a constant number of counters and a constant number of zero-tests. Please write down the amplifier as a counter program.

Exercise 13.2. Let $V = (Q, T)$ be a VASS. Show that if there is a pseudorun (a run over $\mathbb{Z}$) from $(p, \mathbf{0})$ to $(q, \mathbf{0})$ and there exists $\boldsymbol{\Delta} \geq (1, \ldots, 1)$ for which $(p, \mathbf{0}) \xrightarrow{*}_V (p, \boldsymbol{\Delta})$ and $(q, \boldsymbol{\Delta}) \xrightarrow{*}_V (q, \mathbf{0})$, then $(p, \mathbf{0}) \xrightarrow{*}_V (q, \mathbf{0})$.

Exercise 13.3. Let V be a d-VASS whose states and transitions form a single strongly connected component. Suppose that there exists a zero-effect cycle which uses every transition at least once and there are cycles with effects $\boldsymbol{\Delta}_1, \ldots, \boldsymbol{\Delta}_k \in \mathbb{Z}^d$. Show that there exists a cycle of effect $-(\boldsymbol{\Delta}_1 + \ldots + \boldsymbol{\Delta}_k)$ which uses every transition at least once.

Exercise 13.4. Let V be a VASS. Show that if $(p, \mathbf{0}) \xrightarrow{*}_V (p, \boldsymbol{\Delta})$ for some $\boldsymbol{\Delta} \geq (1, \ldots, 1)$; $(q, \boldsymbol{\Delta}') \xrightarrow{*}_V (q, \mathbf{0})$ for some $\boldsymbol{\Delta}' \geq (1, \ldots, 1)$; and, for every $m \geq 1$, there exists a pseudo run (a run over $\mathbb{Z}$) from $(p, \mathbf{0})$ to $(q, \mathbf{0})$ which uses every transition at least m times, then $(p, \mathbf{0}) \xrightarrow{*}_V (q, \mathbf{0})$.

Exercise 13.5. For a given VASS V and configuration c, we define $R_V(c) \coloneqq \{c' : c \xrightarrow{*}_V c'\}$.

Give an example of a d-VASS V along with a configuration $(p, \mathbf{u})$ such that (i) there exists $N \in \mathbb{N}$, such that there does not exist a configuration $(q, \mathbf{v}) \in R_V((p, \mathbf{u}))$ such that $\mathbf{v} \geq (N, \ldots, N)$; and (ii) for every $i \in \{1, \ldots, d\}$, and for every $B \in \mathbb{N}$, there exists a configuration $(r, \mathbf{x}) \in R_V((p, \mathbf{u}))$ such that $\mathbf{x}[i] \geq B$.

Remark. In other words, provide an example of a VASS and a starting configuration such that the VASS is not simultaneously unbounded on all counters but every counter can (perhaps independently) reach an unbounded value.