Infinite Automata 2025/26

Exercise Sheet 7

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Exercise 7.1 *Generalisation of Exercise 6.6.* Show that the set of solutions of systems of integer linear inequalities (over $\mathbb{N}$) is semilinear.

Exercise 7.2 Show that the reachability relation of a linear path scheme (LPS) is semilinear.

Note. The reachability relation of a 2-VASS can be represented as a union of linear path schemes.

Exercise 7.3. Show that semilinear sets are closed under intersection.

Hint. Show that, for every linear set L, there exists a system of integer linear equations with a set distinguished variables such that the set of solutions to the system of integer linear equations projected onto the distinguished set of variables is exactly L.

Exercise 7.4. Show that each semilinear set can be expressed as a union of linear sets whose period vectors are linearly independent.

Hint 1. Show that if a set of periods P is linearly dependent then one can express $b + P^*$ as a finite union of linear sets with fewer periods.

Hint 2. Let P is a linearly dependent set of period vectors; choose a linearly independent subset $Q \subset P$. Every $\mathbf{p} \in P \setminus Q$ can be expressed as a linear combination of elements of Q. Let $\mathbf{p} = a_1\mathbf{q}_1 + \ldots + a_k\mathbf{q}_k$ for some $a_i \in \mathbb{Z}$ and suppose M is an upper bound on such that $|a_1|, \ldots, |a_k|$. Every element of $b + P^*$ can be expressed in such a way that only of the follow is true: $\mathbf{p}$ is not used or there exists $i \in \{1, \ldots, k\}$ such that $\mathbf{q}_i$ is used at most M times. Then one can reduce the number of periods by splitting into each of these $k+1$ cases and removing the period that is used (zero or) a bounded number of times. In the case $\mathbf{q}_i$ is used at most M times, one needs to also split into $M+1$ many cases for each value $x \in [0, M]$ for which $\mathbf{q}_i$ is used x many times.

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