Infinite Automata 2025/26

Exercise Sheet 6

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Exercise 6.1. Show that reachability in d-VASS can be reduced (in logarithmic space) to reachability in $(d+4)$-VAS.

Solution. Let V be the given d-VASS with n states. We encode a state $i \in \{1,\dots,n\}$ as a quadruplet $(i, n-i, 0, 0)$. A transition $(i, \mathbf{x}, j)$ in V is simulated by a transition $(\mathbf{x}, -i, -(n-i), +j, +(n-j))$ in the $(d+4)$-VAS. There are additional transitions $(\mathbf{0}, +i, +(n-i), -i, -(n-i))$ for all $i \in \{1,\dots,n\}$ to restore the encoding.

Exercise 6.2. Show that reachability in d-VASS can be reduced (in logarithmic space) to reachability in $(d+3)$-VAS.

Solution idea. Let V be the given d-VASS with $n-1$ states. We encode a state $i \in \{1,\dots,n\}$ as a triplet $(i, n(n-i), 0)$. A transition $(i, \mathbf{x}, j)$ in V is simulated by three transitions in the $(d+3)$-VAS. Roughly speaking, the three transitions cycle through the following three configurations on the additional three counters in the VAS: $(i, n(n-i), 0) \to (0, j, n(n-j)) \to (n(n-j), 0, j) \to (j, n(n-j), 0)$. At any point, the update $\mathbf{x}$ to the primary d counters can be added. This idea comes from a paper by Hopcroft and Pansiot (1979).

Exercise 6.3. Show that the following sets are not semilinear.

(1) $\{2^n : n \in \mathbb{N}\}$.

(2) $\{2n+1 : n \in \mathbb{N}\} \cup \{2^n : n \in \mathbb{N}\}$.

(3) $\{(n, 2^n) : n \in \mathbb{N}\}$.

(4) $\{(n, m) : n \in \mathbb{N}, m \leq 2^n\}$.

Exercise 6.4. Show that semilinear sets are closed under

(1) union,

(2) projection, and

(3) shift by a vector and then intersecting with $\mathbb{N}^d$.

Exercise 6.5. Show an example of a 3-VASS with semilinear reachability set that is not semilinear. Note: for the reachability sets of any 2-VASS is semilinear, and even its reachability relation is semilinear.

Exercise 6.6. Show that the set of solutions to a system of integer linear equations (over $\mathbb{N}$) with integer coefficients is semilinear.