Infinite Automata 2025/26

Exercise Sheet 5

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Exercise 5.1. Consider an instance of coverability consisting of the VASS V, the initial configuration $(p, \mathbf{u})$, and the target configuration $(q, \mathbf{v})$. Let $B = \{(q, \mathbf{v}) : (p, \mathbf{u}) \xrightarrow{*}_V (q, \mathbf{v})\}$. Show that $B\!\uparrow$ does not satisfy condition (4) of being a coverability separator (Definition 4.8), namely that it is not closed under the transition relation of V.

Solution idea. Consider a two-state 1-VASS consisting of the transition $(p, 1, q)$. It is obvious that $(p, 0)$ can cover $(q, 0)$ but the upward closure of the backwards reachability set from $(q, 0)$ is not a coverability separator.

Exercise 5.2. Consider an instance of coverability consisting of the VASS V, the initial configuration $(p, \mathbf{u})$, and the target configuration $(q, \mathbf{v})$. Let $C = \{c : c \xrightarrow{*}_V (q, \mathbf{v}') \text{ for some } \mathbf{v}' \geq \mathbf{v}\}$. Show that C is a coverability separator (Definition 4.8) for $(V, (p, \mathbf{u}), (q, \mathbf{v}))$.

Solution. See Claim 4.9.

Exercise 5.3. For an ordered set $(X, \leq)$ let $D_{\text{fin}}(X)$ be the set of finite downward closed subsets of X (with respect to the order $\leq$). Show that if $(X, \leq)$ is a WQO then $(D_{\text{fin}}(X), \subseteq)$ is also a WQO.

Exercise 5.4. Show that if a language L over a finite alphabet Σ is upward closed with respect to the subsequence order (if u is in L, then every its supersequence of u is in L) then L is regular.

Hint. Use Higman's Lemma (Exercise 4.7).

Exercise 5.5. For an ordered set $(X, \leq)$ let $D(X)$ be the set of downward closed subsets of X with respect to the order $\leq$. Note that the sets in $D(X)$ need not be finite (in contrast to $D_{\text{fin}}(X)$ in Exercise 5.3). Show that if $(X, \leq)$ is a WQO then there is no infinite descending chain in the order $(D(X), \subseteq)$.

Solution. Consider a sequence $X_1 \supset X_2 \supset \ldots$. For every i, we can extract an element $x_i \in X_i \setminus X_{i+1}$. The infinite sequence $(x_i)_{i=1}^{\infty}$ must satisfy $x_i \not\leq x_j$ for all $i < j$. Otherwise, suppose there are $i < j$ such that $x_i \leq x_j$. It is true that $x_j \in X_j \setminus X_{j+1}$, so $x_j \in X_j$. Since $x_i \leq x_j$ and X_j is downwards closed, $x_i \in X_j$ also. However, we also know that $X_{i+1} \supseteq X_j$ and $x_i \in X_i \setminus X_{i+1} \subseteq X_i \setminus X_j$. This creates a contradiction, so we know that $(x_i)_{i=1}^{\infty}$ satisfies $x_i \not\leq x_j$ for all $i < j$. This is a contradiction because $(X, \leq)$ is a WQO and so there cannot be an infinite descending chain in $D(X)$.

In the following four exercises, we will use the definition $R_V(c) \coloneqq \{c' : c \xrightarrow{*}_V c'\}$.

Exercise 5.6. Find a 3-VASS $V = (Q, T)$ and a configuration $(p, \mathbf{u}) \in Q \times \mathbb{N}^3$ such that the cardinality of $R_V((p, \mathbf{u}))$ is at least doubly exponential with respect to the size of V encoded in binary.

Exercise 5.7. Find a 3-VASS $V = (Q, T)$ and a configuration $(p, \mathbf{u}) \in Q \times \mathbb{N}^3$ such that the cardinality of $R_V((p, \mathbf{u}))$ is at least k-exponential with respect to the size of V encoded in binary.

Exercise 5.8. Find a 4-VASS $V = (Q, T)$ and a configuration $(p, \mathbf{u}) \in Q \times \mathbb{N}^4$ such that the cardinality of $R_V((p, \mathbf{u}))$ is at least Tower with respect to the size of V encoded in binary.

Exercise 5.9. Find a d-VASS $V = (Q, T)$ and a configuration $(p, \mathbf{u}) \in Q \times \mathbb{N}^d$ such that the cardinality of $R_V((p, \mathbf{u}))$ is at least $\mathcal{F}_{d-1}$ with respect to the size of V encoded in binary.