Infinite Automata 2025/26

Exercise Sheet 4

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Exercise 4.1. Let T be a tree. Prove or disprove: if all paths in T have finite length, then T is finite.

Exercise 4.2. *König's Lemma, specialised to trees.* Let T be a tree. Prove that if T has finite branching (is locally finite) and if all paths in T have finite length, then T itself is finite.

Exercise 4.3. Let X and Y be finite sets and let $\preceq_X$ and $\preceq_Y$ be reflexive and transitive relations over X and Y, respectively, such that $(X, \preceq_X)$ and $(Y, \preceq_Y)$ are well-quasi orders. Let $\preceq$ be a relation over the cartesian product $X \times Y$ such that $(x, y) \preceq (x', y')$ if and only if $x \preceq_X x'$ and $y \preceq_Y y'$. Prove that $(X \times Y, \preceq)$ is a well-quasi order.

Exercise 4.4. *Dickson's Lemma, specialised to the naturals.* Show that $(\mathbb{N}^d, \leq)$ is a well-quasi order where $\mathbf{u} \leq \mathbf{v}$ if and only if $\mathbf{u}[i] \leq \mathbf{v}[i]$, for every $i \in \{1, \ldots, d\}$.

Exercise 4.5. The lexicographic order $\preceq_{\text{lex}}$ is defined by $\mathbf{u} \preceq_{\text{lex}} \mathbf{v}$ if there exists i such that $\mathbf{u}[i] < \mathbf{v}[i]$ and, for all $j \in \{1, \ldots, i-1\}$, $\mathbf{u}[j] = \mathbf{v}[j]$. The set $\mathbb{N}^\omega$ is the set of natural valued vectors of infinite length. Prove or disprove that the following are well-quasi orders.

(1) $(\mathbb{N}^2, \preceq_{\text{lex}})$.

(2) $(\mathbb{N}^d, \preceq_{\text{lex}})$.

(3) $(\mathbb{N}^\omega, \leq)$.

(4) $(\mathbb{N}^\omega, \preceq_{\text{lex}})$.

(5) $(\{a, b\}^*, \preceq_{\text{lex}})$.

(6) $\mathbb{N}$ with divisibility.

(7) Graphs with induced subgraph order.

(8) Trees with induced subgraph order.

Exercise 4.6. *Higman's Lemma, specialised to finite alphabets.* Let Σ be a finite alphabet. Prove that $(\Sigma^*, \preceq)$ is a well-quasi order where $\preceq$ is the subsequence order.

Hint. Consider the smallest counterexample. Then consider the subsequence of words with the same first letter and using this subsequence construct a smaller counterexample. This is a contradiction with the assumption that one selected the smallest counter example.

Exercise 4.7. *Higman's Lemma.* Let X be a not necessarily finite set. Prove that $(X^*, \preceq)$ is a well-quasi order where $\preceq$ is the subsequence order.

Hint. Consider the same proof strategy as Exercise 4.6 but take a subsequence of words such that the first letters form an infinite increasing sequence (Condition (3) of well-quasi orders).

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