Infinite Automata 2025/26

Exercise Sheet 3

Wojciech Czerwiński and Henry Sinclair-Banks

Exercise 3.1. Show that reachability in binary 1-dimensional linear path scheme is in NP.

Exercise 3.2. We define $\text{cone}_{\mathbb{R}}(\mathbf{v_1}, \ldots, \mathbf{v_k})$ as the set of vectors which can be obtained as linear combinations of vectors $\mathbf{v_i}$ with coefficients in $\mathbb{R}_+$.

$$\text{cone}_{\mathbb{R}}(\mathbf{v_1}, \ldots, \mathbf{v_k}) \coloneqq \{n_1\mathbf{v_1} + \ldots + n_k\mathbf{v_k} : n_1, \ldots, n_k \in \mathbb{R}_+\}.$$

Show that if $\mathbf{b} \in \text{cone}_{\mathbb{R}}(\mathbf{v_1}, \ldots, \mathbf{v_k}) \subseteq \mathbb{R}^2$, then there is a subset $S \subseteq \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ of cardinality 2 such that $\mathbf{b} \in \text{cone}_{\mathbb{R}}(S)$. Similarly, for dimension d, show that there there is such a subset of cardinality d. This result is called *Carathéodory's theorem.*

Exercise 3.3. We will now look at analog of Carathéodory's theorem with integer coefficients. We define $\text{cone}_{\mathbb{Z}}(\mathbf{v_1}, \ldots, \mathbf{v_k})$ as the set of vectors which can be obtained as linear combinations of vectors $\mathbf{v_i}$ with natural coefficients.

$$\text{cone}_{\mathbb{Z}}(v_1, \ldots, v_k) \coloneqq \{n_1\mathbf{v_1} + \ldots + n_k\mathbf{v_k} : n_1, \ldots, n_k \in \mathbb{N}\}.$$

Assume that $X \subseteq \mathbb{Z}^d$ is a finite set and suppose $M = \max\{\|x\|_\infty : x \in X\}$. Let $\mathbf{b} \in \mathbb{N}^d$ such that $\mathbf{b} \in \text{cone}_{\mathbb{Z}}(X)$. Show that there is a subset $S \subseteq X$ such that $\mathbf{b} \in \text{cone}_{\mathbb{Z}}(X)$ and $|S| = k$ where $d \leq k \leq d\log(2Mk+1)$.

Hint. Show that if $|S| > k$ then there exist $X, Y \subseteq S$ such that $X \neq Y$ and $\sum_{x\in X} x = \sum_{y\in Y} y$.

Exercise 3.4. Prove that reachability problem for 4-CMs is undecidable.

Solution idea. Reduce from the reachability problem for two-stack PDAs. A two-stack automata can have each of its stacks simulated by a pair of counters. In each pair of counters, one will track the contents of the stack encoded as a binary integer and the other counter will be used to manipulate the value of the other counter such as to help multiply it by two (when pushing to the stack) or divide it by two (when popping from the stack).

Exercise 3.5. Prove that reachability problem for 3-CMs is undecidable.

Solution idea. Notice that in the solution to Exercise 3.4, the "other counters" in each pair can be shared. This means that there would be two counters storing the binary representations of the contents of each of the two stacks and a third, common, auxiliary counter to help manipulate their values.

Exercise 3.6. Prove that reachability problem for 2-CMs is undecidable.

Solution idea. Take the 3-CM from the solution to Exercise 3.5; suppose the counters are x, y, and z. Represent their values on one counter as the value $2^x3^y5^z$ and use the second counter as an auxiliary to manipulate their values. Performing a zero test on counter x, for example, corresponds to checking that $2^x3^y5^z$ is not divisible by 2.

16th October 2025