485 - a new upper bound for Morpion Solitaire

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Abstract. In [2] an upper bound of 705 was proved on the number of moves in the 5T variant of the Morpion Solitaire game. We show a new upper bound of 485 moves. This is achieved in the following way: we encode Morpion 5T rules as a linear program and solve 126912 instances of this program on special octagonal boards. In order to show correctness of this method we analyze rules of the game and use a concept of a potential of a given position. By solving continuous-valued relaxations of linear programs on these boards, we obtain an upper bound of 586 moves. Further analysis of original, not relaxed, mixed-integer programs leads to an improvement of this bound to 485 moves. However, this is achieved at a significantly higher computational cost.

1 Introduction

The Morpion Solitaire is a paper-and-pencil single-player game played on a square grid with the initial configuration of 36 dots depicted in Figure 1. In the 5T variant of the Morpion Solitaire game[1], in each move the player puts a dot on an unused grid position and draws a line that consists of four consecutive segments passing through the dot. The line must be horizontal, vertical or diagonal. None of the four segments used in the line may appear as a segment of any other line. The goal is to find the longest possible sequence of moves.

The problem is notoriously difficult for computers. For 34 years, in the Morpion 5T game the longest known sequence was one of 170 moves discovered by C.-H. Bruneau in 1976 (see[2] [1]). The record was finally broken by Christopher D. Rosin, who presented in [6] a configuration of 177 moves, obtained using a Monte Carlo algorithm called Nested Rollout Policy Adaptation (NRPA). In 2011 (see [1]) he improved his record to 178, which is the best result known today. The webpage [1] maintained by Christian Boyer, contains an extensive and up-to-date information about records in all Morpion Solitaire variants.

As the Morpion 5T game is played on a potentially infinite grid, a priori it is not clear whether the maximal sequence should be finite at all. An upper bound of 705 was shown in [2]. In the present paper we show a new upper bound

[1] We refer to this variant as the Morpion 5T game. For an overview of other variants see the webpage [1] or the paper [2].

[2] Regarding this and other records we refer to the webpage [1] for a detailed description and further references.

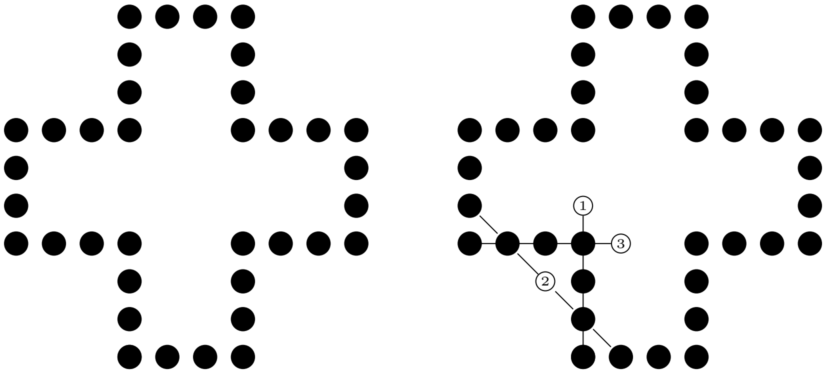

Fig. 1: Initial position of Morption 5T on the left and a position with 3 moves on the right.

of 485. We base our approach on the observation that a Morpion 5T game may be expressed as a mixed-integer linear programming problem.

We observed (see Section 2.1) that the bound obtained by a continuous-valued linear relaxation of the Morpion 5T game significantly depends on the size of the grid on which the game is played. On big grids the bound may be well over the upper bound of 705 (in fact, we do not know if it is finite). On the other hand on smaller grids we obtained useful bounds. For example, on a grid limited to a regular octagon[3] with sides of length 10, we obtain a linear relaxation bound of 543. By a geometric reasoning, using the notion of potential and careful analysis of the rules of Morpion 5T game, we show that every Morpion 5T position must be contained in one of 126912 octagonal grids with small boundaries (see Lemma 4 for a precise formula). The maximum bound of 586.82 is obtained on a grid which is an octagon with sides of lengths 10, 8, 10, 12, 10, 8, 10, 12 (see Figure 5).

In fact, the picture is more complicated. To state a linear problem for Morpion 5T we need not only to consider the shape of the octagonal grid but also a position of the initial cross inside. This makes the number of cases to consider larger by two orders of magnitude. To get around this difficulty we consider variants of Morpion Solitaire called Morpion 5T+ and Morpion 5T++ (see [1]). In the later variant the position of the initial cross inside of the grid is not relevant. Every Morpion 5T game is also a Morpion 5T+ game and every Morpion 5T+ game is a Morpion 5T++ game. The difference between 5T and 5T+ is that the line drawn in a move needs not to pass through the dot placed in this move. The difference between 5T+ and 5T++ is that one may place more than one dot in a single move, as long as in the final position the number of dots is equal to the number of moves plus 36. In other words, we may borrow dots in 5T++ as long as the balance at the end is correct. In Morpion 5T++ we start with an empty board.

[3] This is a graph that plays a role in the proof of the 705 bound in [2].

The upper bound of 705 moves proved in [2] is also valid in the Morpion 5T++ game. The longest known sequence of moves in Morpion 5T+ was found by Marc Bertin in 1974 and consists of 216 moves (see [1]). The longest known sequence of moves in the Morpion 5T++ game was found by Christian Boyer in 2011 (see [1]). The sequence consists of 317 moves.

The associated mixed-integer linear problems are much easier to solve in the case of 5T++ variant and we have a benefit of much smaller number of cases to consider. However, the limit on the size of the grid pertains to the 5T variant, so our new upper bound of 485 is valid for the 5T variant only.

The paper is organized as follows. In Section 2 we formulate the linear problem (LP0)—(LP3). In Section 3 we calculate that the number of instances, which must be treated by the solver, is 126912 . In Section 4 we consider consequences of the relaxation of the original problem (LP0)—(LP3). This allows to show an upper bound of 586 moves in the Morpion 5T game. In Section 5 we push the result of Section 4 in order to obtain an upper bound of 485 moves. This is done at a considerable increase in the computation time. In Section 6 we explain the correctness of algorithms used in previous Sections. The correctness result boils down to an observation how, in terms of potential, a given board relates to the smallest octagon containing this board (see Theorem 1).

We note that modern LP solvers have no problem in finding the record sequence of 317 moves for the 5T++ variant, but despite considerable computational effort we were not able to break this record. The current upper bound of 485 can be improved with more computational resources. However, we believe that the best approach would be to find better limitations on the size of grids.

It is also possible to write linear programs that solve the Morpion 5D variant of the Morpion Solitaire (see [1] for description of the rules and current lower and upper bounds). On larger grids we obtain objective 144 for the relaxed problem, as the standard potential-based argument applies to the relaxed case as well (see[4] [2]). The upper bound of 121 moves in the Morpion 5D game was proved in [5]. Using variants of the Morpion 5D game and a different strategy of limiting grids, we were able to prove that an upper bound in the Morpion 5D game is below 100. We also proved that the best possible result in the symmetric Morpion 5D game is 68. These results will be presented in a separate publication.

2 Linear Relaxation

A *lattice point* on a plane is a point with integer coordinates. A *lattice graph* is a graph with vertices in lattice points and edges consisting of pairs (p, q), where p and q are two different neighboring points, that is $p \neq q$ and $p = (n, m)$ and $q = (n \pm i, m \pm j)$ for some $i, j = 0, 1$. We call such edges the *lattice edges*.

A *move* in a lattice graph $G = (V, E)$ is a set of four consecutive parallel lattice edges. We let $\mathcal{M}(G)$ to be the set of all moves in a graph G. We start with the following observation, which simply rephrases the rules of Morpion 5T++ formulated in the Introduction.

[4] In fact, applying additional argumentation, in [2] is shown a bound of 141 moves.

Lemma 1. *A graph $G = (V, E)$ is a Morpion 5T++ position graph if and only if it satisfies the following conditions*

(M1) G is a lattice graph,
(M2) $4 \cdot \#V - \#E = 144$,
(M3) The set E of edges of G can be decomposed into a collection of disjoint moves.

Let $B = (V_B, E_B)$ be a fixed lattice graph that we shall call *the board*. In applications, it will be a sufficiently large octagonal lattice graph with a full set of edges. Below we define linear constraints that describe all subgraphs of B that satisfy conditions (M1)— (M3) of Lemma 1.

We define the following set of structural *binary* variables, that is variables assuming values $0, 1$:

$$\{\mathrm{dot}_v \colon v \in V_B\} \cup \{\mathrm{mv}_m \colon m \in \mathcal{M}(B)\}. \tag{LP1}$$

For each $e \in E_B$ and $v \in e$ we declare the following constraints:

$$\sum_{m \in \mathcal{M}(B), e \in m} \mathrm{mv}_m \leq \mathrm{dot}_v\,. \tag{LP2}$$

$$\sum_{v \in V_B} \mathrm{dot}_v = 36 + \sum_{m \in \mathcal{M}(B)} \mathrm{mv}_m\,. \tag{LP3}$$

The following two lemmas describe correspondence between binary-valued solutions of a mixed integer programming problem (LP1) - (LP3) and subgraphs of B that are Morpion 5T++ positions.

Lemma 2. *Let $G = (V_G, E_G)$ be a subgraph of B and a Morpion 5T++ position obtained by a sequence $\mathcal{M}$ of moves. If*

$$\mathrm{dot}_v = \begin{cases} 0 & \textit{if } v \notin V_G \\ 1 & \textit{if } v \in V_G \end{cases} \quad \textit{and} \quad \mathrm{mv}_m = \begin{cases} 0 & \textit{if } v \notin \mathcal{M} \\ 1 & \textit{if } v \in \mathcal{M} \end{cases},$$

then conditions (LP1), (LP2) and (LP3) hold.

Proof. If $\mathrm{dot}_v = 0$, then there is no move passing through v, hence the left hand side of (LP2) is equal to 0. If $\mathrm{dot}_v = 1$, then condition (LP2) means that every segment e played in the game can appear in exactly one move. Condition (LP3) means that the number of dots placed is higher by 36 than the number of moves made.

Lemma 3. *Assume that a set of variables defined by condition (LP1) satisfies conditions (LP2) and (LP3). Let $G = (V_G, E_G)$ be a graph with a set of vertices*

$$V_G = \{v \in V_B \colon \mathrm{dot}_v = 1\}$$

and a set of edges

$$E_G = \{e \in E_B \colon \exists_{m \in \mathcal{M}(B)}\ e \in m, \mathrm{mv}_m = 1\}.$$

Then G is a Morpion 5T++ position and a subgraph of G.

Proof. We will show that G satisfies conditions (M1) — (M3) of Lemma 1.

By the definition of E_G, if $e \in E_G$ then there exists $m \in \mathcal{M}(B)$ such that $\mathrm{mv}_m = 1$. By (LP2), if $\mathrm{mv}_m = 1$, then $\mathrm{dot}_v = 1$ for each $v \in V_B$ such that $v \in e \in m$. It means that graph G contains vertices of its edges, therefore it is a well defined subgraph of B, hence it is a lattice graph and it satisfies (M1).

From (LP2) follows, that the moves mv_m must be disjoint in the sense, that they cannot contain the same edge twice. This implies condition (M3) of Lemma 1. From disjointness and condition (LP3) follows condition (M2) of Lemma 1.

We consider a linear relaxation of the MIP problem (LP1)—(LP3). We let structural variables to be real-valued, subject to bounds

$$0 \leq \mathrm{dot}_v, \mathrm{mv}_m \leq 1. \tag{LP4}$$

In the relaxation we maximize the objective function

$$\sum_{m \in \mathcal{M}(B)} \mathrm{mv}_m \tag{LP0}$$

Clearly, an optimal solution to the linear programming problem (LP0) - (LP4) gives an upper bound for the length of a Morpion 5T++ game on a board B.

2.1 The problem of infinite grid

Observe that any lattice graph that consists of 9 vertex-disjoint moves has 45 vertices and 36 edges and satisfies conditions (M1) — (M3) of Lemma 1, hence it is a Morpion 5T++ position graph and consequently Morpion 5T++ positions can have arbitrarily large diameter in the plane $\mathbb{R}^2$.

The following table summarizes solutions of the linear relaxation of Morpion 5T++ on square $n \times n$ boards (where n is the number of edges on the side).

10	20	30	40	50	60	70	80	90	100
64.00	278.50	619.53	876.55	1130.01	1387.54	1641.74	1898.13	2152.86	2408.54

Fig. 2: The top row contains the length n of the edge of a given square and the bottom row contains solutions to the relaxed problem (LP0)—(LP4) on the $n \times n$ board.

We do not know whether the objective function (LP0) is bounded or not on the infinite grid. However, the bound of 705 moves derived in [2] holds for Morpion 5T++. This shows that we get no useful upper bound for positions satisfying (M1) — (M3) using our linear relaxation method. To get a bound, we have to use another properties of Morpion 5T positions to bound the size of the board. This will be done in the next Section.

3 Bounding the Board

Let $G = (V, E)$ be a lattice graph. Following [2] we define the *potential of G*

$$\text{potential}(G) = 8 \cdot \#V - 2 \cdot \#E.$$

In this Section it will appear that the missing constraint which makes the original linear problem (LP0—LP4) accessible to modern LP solvers is an additional bound on the shape and potential of the board (see Theorem 1), which in turn, thanks to Lemma 4, will imply a bound on the size of relevant boards. In order to formulate the bound we need some new geometric notions.

A *half–plane graph* is a full lattice graph with a vertex set of all lattice points $\langle x, y\rangle$ such that

$$ax + by + c \geq 0$$

where $a, b \in \{-1, 0, 1\}$ with $a \neq 0$ or $b \neq 0$ and $c \in \mathbb{Z}$. We define $\mathcal{H}$ as the set of all half-plane graphs.

An *octagonal hull* of a lattice graph $G = (V_G, E_G)$, denoted $\text{hull}(G)$, is an intersection of all half–plane graphs containing G, i.e.

$$\text{hull}(G) = \bigcap \{H = (V_H, E_H) \colon V_G \subset V_H, E_G \subset E_H, H \in \mathcal{H}\}.$$

We call octagonal hulls *octagons*.

Every octagon has eight edges. We may describe octagon by giving lengths of its edges. We start with the top edge and continue clockwise. For example, octagon depicted in Figure 3 has edges of lengths 3, 3, 0, 1, 6, 0, 3, 1. We call every diagonal edge of length 0 a corner of an octagon. Corners are opposite if they correspond to parallel edges of the octagon.

In the next two Sections we will obtain an upper bound of 586 and respectively 485 moves for Morpion 5T game solving 126912 instances of the linear problem (LP0)—(LP4) described in Section 2. The following Theorem shows that the penalty, measured in extra potential, paid for solving such problems only on octagonal hulls is relatively small. In turn, thanks to Lemma 4, the bound on the potential allows to limit the size of octagons. This will allow us to focus attention on 126912 relatively small octagonal instances of linear programs. The number 126912 will be deduced in Theorem 2 later in this Section.

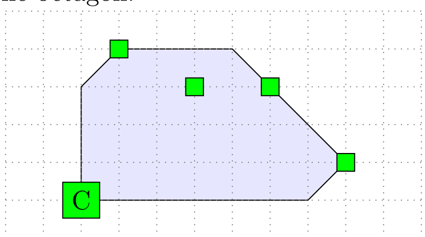

Fig. 3: The octagonal hull of green points. The green point marked with the letter C is a corner *of the octagon hull.*

Theorem 1. *Let G be a Morpion 5T position graph.*

1. *If* $\text{hull}\, G$ *does not contain any corner, then*

$$\text{potential}(\text{hull}(G)) \leq \text{potential}(G)$$

2. If it does not contain opposite corners, then

$$\text{potential}(\text{hull}(G)) \leq \text{potential}(G) + 2$$

3. If it contains at least two opposite corners, then

$$\text{potential}(\text{hull}(G)) \leq \text{potential}(G) + 4.$$

Let modifier(hull(G)) *by equal to* $0, 2, 4$ *like in the three above cases.*

We postpone the proof of Theorem 1 until Section 6.

3.1 The set of boards

In this section we describe a set of octagonal boards that contain every possible Morpion 5T position. As this set is quite large, we will use symmetry to limit its size.

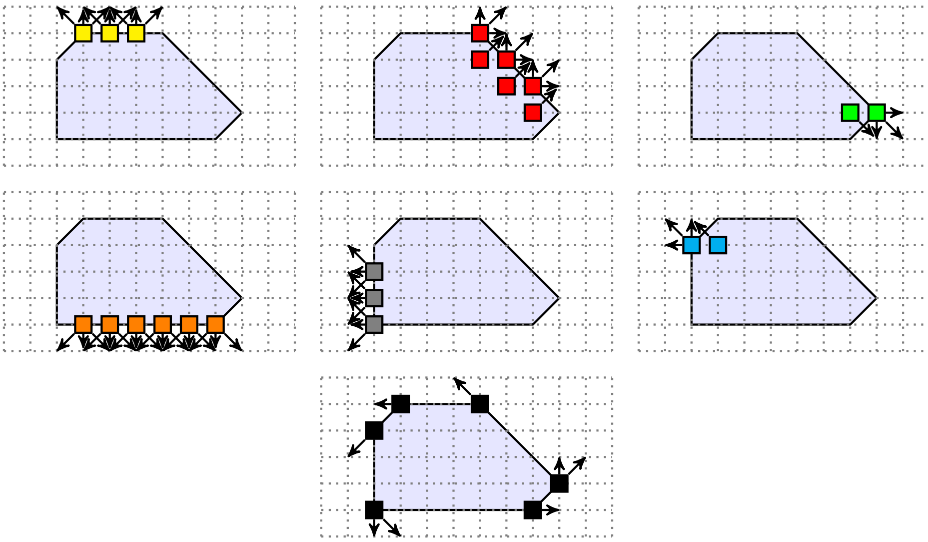

Fig. 4: The octagon $(3, 3, 0, 1, 6, 0, 3, 1)$*.The top 6 figures represent potential associated with* $3a_1 = 3 \cdot 3 = 9$*,* $4a_2 = 4 \cdot 3 = 12$*,* $4a_4 = 4 \cdot 1 = 4$*,* $3a_5 = 3 \cdot 6$*,* $3a_7 = 3 \cdot 3 = 9$*,* $4a_8 = 4 \cdot 1 = 4$*. The bottom figure represents missing* 8 *edges of the potential.*

We say that a graph G is *non-degenerated* if it contains three vertices that are not on a single diagonal line.

Lemma 4. *Let* G *be a non-degenerated octagon with edge lengths* $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ *with* a_1 *denoting the length of the top edge. The following equations hold.*

$$\text{potential}(G) = 8 + 3a_1 + 4a_2 + 3a_3 + 4a_4 + 3a_5 + 4a_6 + 3a_7 + 4a_8. \qquad \text{(O1)}$$

$$a_8 = a_2 + a_3 + a_4 - a_7 - a_6. \tag{O2}$$

$$a_1 = a_4 + a_5 + a_6 - a_8 - a_2. \tag{O3}$$

If G contains the starting cross of Morpion 5T game, then

$$a_1 + a_2 + a_3 \geq 10 \tag{O4}$$

$$a_8 + a_1 + a_2 \geq 10 \tag{O5}$$

$$a_2 + a_3 + a_4 \geq 10 \tag{O6}$$

Using rotation by multiple of 90 *degrees and a reflection along the y-axis we can always obtain a graph that satisfies*

$$a_1 \geq a_3, a_1 \geq a_5, a_1 \geq a_7 \tag{O7}$$

$$a_8 \geq a_2 \tag{O8}$$

Proof. Condition (O1) follows from distribution of potential visualized in Figure 4. Conditions (O2), (O3) are elementary geometric properties of octagons. We will verify them on the example presented in Figure 4. Indeed,

$$a_8 = 1.$$

On the other hand

$$a_2 + a_3 + a_4 - a_7 - a_6 = 3 + 0 + 1 - 3 - 1 = 1.$$

Similarly,

$$a_1 = 3$$

and

$$a_4 + a_5 + a_6 - a_8 - a_2 = 1 + 6 + 0 - 1 - 3 = 3.$$

Properties (O4),(O5),(O6) follows from the observation that in order to embed the starting cross of Morpion 5T game (see Figure 1), the projections of the octagon in diagonal, horizontal and vertical directions must be long enough.

Property (O7) can be guaranteed through rotation by a multiple of 90 degrees. Then property (O8) can be guaranteed through reflection along the y–axis. This reflection preserves property (O7).

Theorem 2. *Let $\mathcal{O}$ denote the set of octagons O that satisfy constraints O1 — O8 of Lemma 4 and the constraint* $\text{potential}(O) = 288 + \text{modifier}(O)$*. The number of elements of $\mathcal{O}$ is* 126912 *and the octagon with the largest number of vertices is an equilateral octagon with sides of length* 10*. This octagon contains* 741 *vertices.*

Proof. Every octagon O with $\text{potential}(O) < 288 + \text{modifier}(O)$ is included in an octagon O' with $\text{potential}(O') = 288 + \text{modifier}(O')$. Hence we can ignore in our calculations octagons with $\text{potential}(O) < 288 + \text{modifier}(O)$. The number of relevant octagons calculated using the script `octagons.cpp` (see the repository [4]). The script generates all instances of octagons satisfying constraints of this Theorem.

As a corollary we obtain the bound presented in [2].

Corollary 1 ([2]). *The number of moves in a Morpion 5T game is bounded by 705.*

Proof. We list all octagons in $\mathcal{O}$ and check how many dots can placed in a given octagon, given the starting 36 dots. The best result consists of 705 new dots for the equilateral octagon with sides of length 10.[5]

Let us notice, that this Corollary is weaker than the one obtained in [2], because the method does not allow to deal with Morpion 5T++ boards.

4 Linear Relaxation Bound

Theorem 3. *Let* obj_B *denote the value of optimal solution of a linear programming problem (LP0) - (LP3). If* B *satisfies conditions (O1) - (O8) of Lemma 4, then*

$$\mathrm{obj}_B \leq 586.82353.$$

The maximum value is obtained for the octagon

$$B = (10, 8, 10, 12, 10, 8, 10, 12).$$

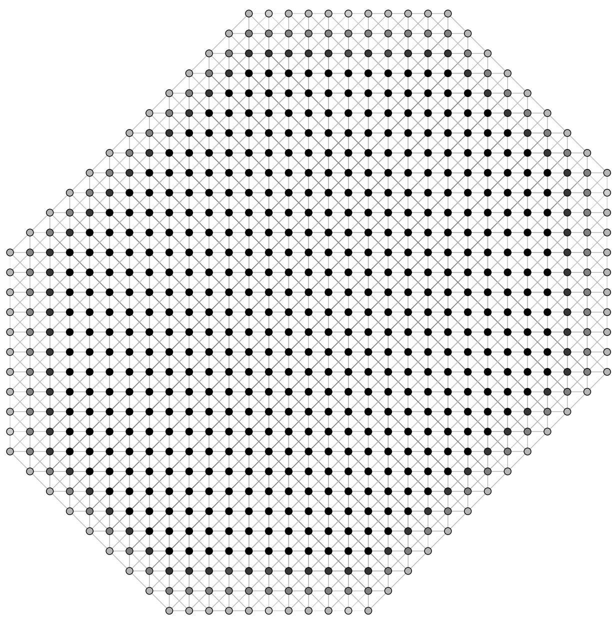

Fig. 5: The record illustrating maximal solution of the relaxed problem. This solution is obtained for the octagon $(10, 8, 10, 12, 10, 8, 10, 12)$*. Since this is a relaxed problem (LP0)—(LP4), the grayness in the Figure indicates the value of the move, that is a number between* 0 *and* 1*.*

[5] This proof does not rely on linear optimization. We just go over a finite list of octagons.

The record solution can be found in Figure 5. All 126912 relaxed problems were solved by `gurobi` optimization software (see [3]) within 24 hours on a single core of a Linux machine equipped with `Intel`® `Xeon`® `CPU X3220@2.40GHz` with 8GB of RAM.

Proof. The proof is a calculation of obj_B for all octagons satisfying conditions O1 — O8 of Lemma 4. The source code of the program `generator.cpp` generating the relevant linear programs can be found in the repository [4].

Corollary 2. *The number of moves in a Morpion 5T game is bounded by* 586.

5 MIP bound: 485

Let us notice that results obtained in Theorem 3 can be naturally strengthened through longer computations. In practice, realistic value we were able to target within two months of computations with three off–the–shelf Linux PC was the objective 485.

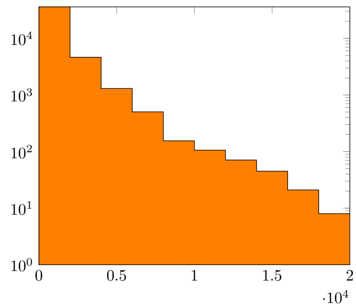

Fig. 6: The vertical axis shows the number of cases and the horizontal axis shows the computation time in seconds.

From Section 4 we know all 126912 instances and their performance under relaxed (LP0)—(LP4) linear problem. Apparently, out of 126912 problems 42889 instances have the relaxed bound bigger or equal to 485 . These are exactly the instances which must be treated by direct computations if we want to reduce the bound to 485 . The total computation time for this target was approximately 310 days using the optimization software `gurobi` (see [3]) on a single core of a Linux machine equipped with `Intel`® `Xeon`® `CPU X3220@2.40GHz` with 8GB of RAM. The graph 6 shows on the logarithmic scale the distribution of the computation time among 42889 instances[6].

6 Geometry of the problem

In this Section we will show a proof of Theorem 1. The key technical ingredient is Lemma 10. Let $\mathcal{M} = \mathcal{M}(G)$ be the set of all possible moves in a graph G. A lattice graph with a vertex set V is *full* if its edge set is maximal, i.e.

$$E = \{(u, v) \colon u, v \in V, u \neq v, |u_x - v_x| \leq 1, |u_y - v_y| \leq 1\}.$$

A graph $G = (V, E)$ is 1-*connected* if a full lattice graph with a vertex set V is connected. A *boundary* of a lattice graph $G = (V, E)$ is a set

$$\mathcal{B}(G) = \{\langle u, v\rangle \in \mathbb{Z}^2 \times \mathbb{Z}^2 \colon u \in V, (u, v) \notin E\}.$$

[6] In fact, a half of the instances required less than 100 sec. to reach the limit of 485 and 9 instances required the computation time longer than 18000 seconds.

Observe that elements of $\mathcal{B}(G)$ are directed edges with start points in V such that the corresponding undirected edge is not in E. Let us notice, that

$$\text{potential}(G) = \#\mathcal{B}(G).$$

where potential(G) is the number defined at the beginning of Section 3. Here we will analyze the potential more closely and divide it into *external* and *internal* potentials.

An edge $e = \langle u, v \rangle \in \mathcal{B}(G)$ is an *external edge* if $u + k \cdot (v - u) \notin V$ for every $k \geq 1$. Let $\mathcal{B}^{\text{ex}}(G)$ denote the set of all external edges of G.

An edge $e \in \mathcal{B}(G)$ is an *internal edge* if it is not an external edge. Let $\mathcal{B}^{\text{int}}(G)$ denote the set of all internal edges of G.

The *external potential* potential$^{\text{ex}}(G)$ is the cardinality of the set $\mathcal{B}^{\text{ex}}(G)$. The *internal potential* potential$^{\text{int}}(G)$ is the cardinality of the set $\mathcal{B}^{\text{int}}(G)$.

Lemma 5. *If G is a Morpion position graph, then* potential$(G) = 288$.

Proof. It follows From the definition of potential at the beginning of Section 3 we have

$$\text{potential}(G) = 8\#V - 2\#E.$$

The number 288 for Morpion position graphs follows from property (M2) in Lemma 1.

In the proof of Theorem 1 we need some additional definitions.

Definition 1. *Let $\mathcal{L}$ denote the*

$$\mathcal{L} = \{l_{a,b,c} \colon (a, b) \in \mathcal{D}, c \in \mathbb{Z}\}.$$

where

$$l_{a,b,c} = \{(x, y) \in \mathbb{Z}^2 \colon ax + by + c = 0\}$$

A line $l_{a,b,c} \in \mathcal{L}$ is called diagonal if $\langle a, b \rangle \in \{-1, 1\}^2$. A graph $G = (V, E)$ is degenerated *if there exists a line $l \in \mathcal{L}$ such that $V \subset l$.*

Definition 2. *A line $l \in \mathcal{L}$ is a* gap line *for graph G if l is diagonal, does not contain any vertices of G, but there are vertices of G on both sides of l. We let* gap(G) *be the number of gap lines of a graph G.*

Lemma 6. *If a lattice graph $G = \langle V, E \rangle$ is an octagon, then for every $l \in \mathcal{L}$ the intersection $V \cap l$ is 1–connected and* potential$(G) =$ potential$^{\text{ex}}(G)$.

Proof. For every $l_{a,b,c} \in \mathcal{L}$ the intersection in $\mathbb{R}^2$ of $\{(x, y) \in \mathbb{R}^2 : ax+by+c = 0\}$ with the convex hull of G in $\mathbb{R}^2$ is an interval in $\mathbb{R}^2$ and the lattice points of this interval are 1–connected and coincide with $V \cap l$.

Lemma 7. *If G is not degenerated, then* gap(hull$(G)) = 0$.

Proof. Fix a diagonal line l. We will show that l is not a gap line for $\mathrm{hull}(G)$. Take vertices $u, v \in \mathrm{hull}(G)$ on both sides of l. Since G is non-degenerated, we may assume that u and v are not on a same line perpendicular to l (first we select arbitrary two points on both sides of l and if they are located on the same line then from degeneracy we can find another either on the side of u or on the side of v).

From the definition, $\mathrm{hull}(G)$ contains the intersection of all half-planes that contain u and v, hence it contains a parallelogram with the following characteristics:

– opposite vertices of the parallelogram are u and v,
– a pair of edges of parallelogram is horizontal (or vertical)
– another pair of edges is diagonal (see Figure 7).

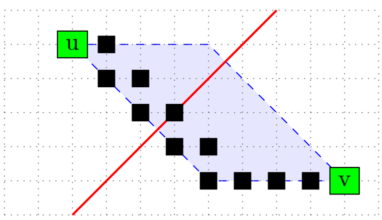

Fig. 7: Every line parallel to l with u and v on different sides must contain one of the vertices marked with black squares.

We start from the vertex v and mark the black dots along the horizontal/vertical edge. Then along the diagonal edge we mark "the staircase" as in Figure 7. Notice, that l passes through one of the black dots of the staircase. Indeed, since u and v are not on a line perpendicular to l, if l intersect the parallelogram between two points on the diagonal edge, then l necessarily pass through a black dot between these two points.

If l intersects one of the horizontal/vertical edges of the parallelogram, then l passes through one of the black dots located on the edge. Hence l has a non–empty intersection with $\mathrm{hull}(G)$ and it follows that l is not a gap line for $\mathrm{hull}(G)$.

The following lemma provides a sufficient condition for a line in $\mathcal{L}$ to be a gap line.

Lemma 8. *Let G be a 1-connected, bounded and non-degenerated lattice graph and $l \in \mathcal{L}$. If l does not contain a vertex of G and contains a vertex of* $\mathrm{hull}(G)$*, then l is a gap line for G.*

Proof. Every half-plane graph that contains l must contain at least one vertex of G, since l has a vertex in $\mathrm{hull}(G)$ (otherwise the opposite half-plane with l removed would contain whole G and hence also $\mathrm{hull}(G)$, so $\mathrm{hull}(G)$ would be disjoint from l). Therefore both half-planes that have l as a boundary contain vertices of G. Since G is disjoint with l, there are vertices of G on both sides of l. In order to prove that l is a gap line it is enough to verify that l is diagonal.

Indeed, observe that horizontal and vertical lines disconnect the grid $\mathbb{Z}^2$ into two 1-connected components. Since G contains vertices on both sides of l and is 1-connected, the line l must be diagonal, hence it is a gap line for G.

The above Lemma 8 along with Lemma 7 shows a characterization of gap lines among lines in $\mathcal{L}$.

Lemma 9. *If G is a 1-connected, non–degenerated lattice graph, then*

$$\text{potential}(\text{hull}(G)) = \text{potential}^{\text{ex}}(G) + 2\,\text{gap}(G).$$

Proof. In this proof it will be convenient to mark as $\overline{e}$ the set consisting of two vertices at the ends of a given edge e. Observe that for any graph Γ and any line $l \in \mathcal{L}$

$$\#\{\overline{e}\colon e \in \mathcal{B}^{\text{ex}}(\Gamma),\ \overline{e} \subset l\} \tag{*}$$

is 0 iff $V_\Gamma \cap l = \emptyset$ and 2 otherwise. By Lemma 6,

$$\text{potential}(\text{hull}(G)) = \text{potential}^{\text{ex}}(\text{hull}(G)).$$

We have

$$\text{potential}^{\text{ex}}(\text{hull}(G)) = \sum_{l\in\mathcal{L}} \#(l \cap \mathcal{B}^{\text{ex}}(\text{hull}(G)))$$

and

$$\text{potential}^{\text{ex}}(G) = \sum_{l\in\mathcal{L}} \#(l \cap \mathcal{B}^{\text{ex}}(G)).$$

By (*) and by Lemma 8, for a given $l \in \mathcal{L}$ either $\#(l \cap \mathcal{B}^{\text{ex}}(\text{hull}(G))) = \#(l \cap \mathcal{B}^{\text{ex}}(G))$ or l is a gap line for G and $\#(l \cap \mathcal{B}^{\text{ex}}(\text{hull}(G))) = 2$, $\#(l \cap \mathcal{B}^{\text{ex}}(G)) = 0$. Hence

$$\text{potential}(\text{hull}(G)) = \text{potential}^{\text{ex}}(G) + 2\,\text{gap}(G).$$

The main technical difficulty in the Section is the following geometric Lemma. This Lemma together with Lemma 9 finish the proof of Theorem 1.

Lemma 10. *If G is a position of the Morpion 5T then*

$$2\,\text{gap}(G) \leq \text{potential}^{\text{int}}(G) + \text{modifier}(\text{hull}(G)).$$

The notion of modifier was defined in Theorem 1.

Proof. Let $G = (V, E)$. Let $\mathcal{L}(G)$ denote the set of all gap lines of G. Let $l \in \mathcal{L}(G)$. The two halfplanes bounded by l decompose the set V of vertices of G into two disjoint subsets, one of which contains all dots of the initial cross. If the other set contains only a single vertex, then we say that l is a singular gap line. Otherwise we say that l is a non-singular gap line. If l is a singular gap line, then we let v_l denote the single vertex of V separated from the initial cross by line l and call v_l the singular vertex of l.

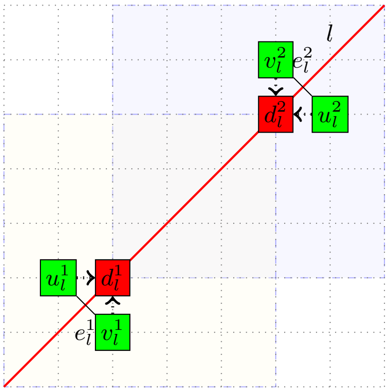

Fig. 8: Green vertices belong to the graph, red vertices do not belong to the graph. From the dotted arrows we will choose 2 to compensate for the gap line l.

Let $m_1, m_2, \ldots, m_n$ be a sequence of the Morpion 5T moves that lead to a position G. Let l be a singular gap line and let m_k be a move that puts the singular vertex v_l on board. Since there are no other vertices in the halfplane bounded by l that contains v_l, no move in sequence $m_1, m_2, \ldots$ requires dot v_l and the sequence $m_1, m_2, \ldots, m_{k-1}, m_{k+1}, \ldots, m_n$ is a valid move sequence. Likewise $m_1, m_2, \ldots, m_{k-1}, m_{k+1}, \ldots, m_n, m_k$ is valid. Hence we may modify our sequence so that the moves that put singular dots on board are at the very end of the move sequence.

Let $m_1, m_2, \ldots, m_k, m_{k+1}, \ldots, m_n$ be a sequence of Morpion 5T moves that lead to a position G such that moves $m_1, m_2, \ldots, m_k$ put non-singular dots on board and moves $m_{k+1}, \ldots, m_n$ put singular dots on board. Let $H = (V_H, E_H)$ be a Morpion 5T position obtained by a sequence $m_1, m_2, \ldots, m_k$. We will show that

$$2\,\mathrm{gap}(H) \leq \mathrm{potential}^{\mathrm{int}}(H).$$

Observe that H has no singular gap lines as removing a singular vertex cannot make a non-singular vertex singular.

Let $l \in \mathcal{L}(H)$. Since l is non-singular and H is obtained as a position in Morpion 5T game, there are two edges $e_l^1, e_l^2 \in E_H$ that cross l. Consider labeling of dots as in Figure 8. Note that d_l^1 and d_l^2 are picked on l between e_l^1 and e_l^2 (and they may be the same point when e_l^1 and e_l^2 are next to each other).

We will construct a map that assigns to each e_l^i ($i = 1, 2$, $l \in \mathcal{L}(H)$) one edge from the list

$$(u_l^i, d_l^i), (v_l^i, d_l^i) \qquad (**_l^i)$$

in such a way that the following conditions are satisfied.

1. The assigned edges realize the internal potential of H, i.e. they belong to $\mathcal{B}^{\mathrm{int}}(H)$.
2. We do not assign the same edge twice.

First we'll show that at least one edge from the edge list $(**_l^i)$ belongs to $\mathcal{B}^{\mathrm{int}}(H)$. Without a loss of generality we may assume that $i = 1$. Consider two half-lines starting at d_l^1 in directions (u_l^1, d_l^1) and (v_l^1, d_l^1) (the dotted arrows in Figure 8). They disconnect the grid of lattice points into two 1-connected components. Both components contain vertices of H (e.g. v_l^1 and v_l^2 are in different components). Since H, as a Morpion 5T position, is 1-connected, there must be a vertex of H on at least one of those half-lines. Since d_l^1 does not belong to V_H (as $d_l^1 \in l$ and l is disjoint from V_H as a gap line), at least one of the edges $(u_l^1, d_l^1), (v_l^1, d_l^1)$ belongs to $\mathcal{B}^{\mathrm{int}}(H)$.

Second, we'll show how to pick edges from the edge list $(**_l^i)$ in such a way that the assignment is unique (one-to-one).

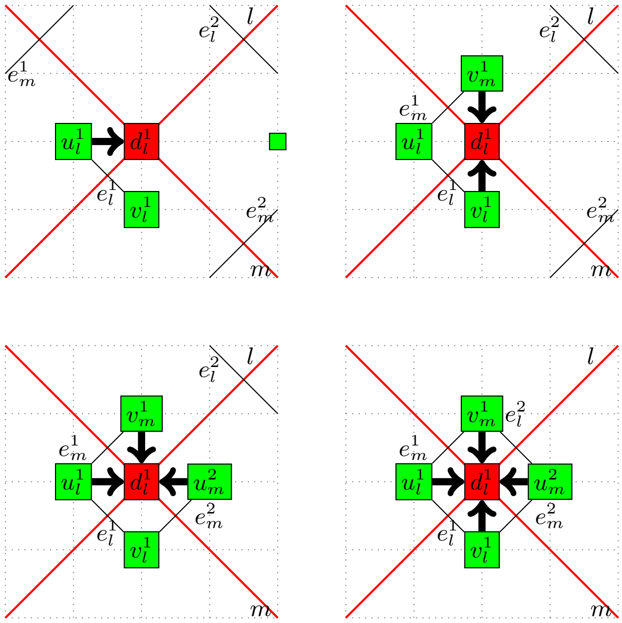

Fig. 9: Two gap lines l and m intersecting at a point d_l^1. Four figures relate to four cases in the proof. The top left is related to Case 1, the top right to Case 2, the bottom left to Case 3 and the bottom right to Case 4.

Consider edge e_l^1 that crosses a gap line l. There may be only one gap line m such that the edge lists $(**_m^1)$ or $(**_m^2)$ overlap with the edge list $(**_l^i)$. We consider four cases about how edges e_l^2, e_m^1, e_m^2 are placed around e_l^i.

Case 1. If both e_m^1 and e_m^2 are vertex disjoint from e_l^1, then we assign to e_l^1 any edge from $(**_l^1)$ that belongs to $\mathcal{B}^{\text{int}}(H)$.

Case 2. Exactly one of e_m^1 and e_m^2 has a common vertex with e_l^1. Without a loss of generality we may assume that e_m^1 has a common vertex $u_l^1 = u_m^1$ with e_l^1. We must be careful to not assign edge (u_l^1, d_l^1) to both edges e_l^1 and e_m^1. We assign (v_l^1, d_l^1) to e_l^1 and (v_m^1, d_m^1) to e_m^1.

Case 3. Both e_m^1 and e_m^2 have a common vertex with e_l^1 but e_l^2 is vertex disjoint from e_m^1 and e_m^2. Assume that v_m^1 and u_m^2 are vertices of e_m^1 and e_m^2 that are disjoint from e_l^i. We assign (v_m^1, d_m^1) to e_m^1, (u_m^2, d_m^2) to e_m^2 and (u_l^1, d_l^1) to e_l^1.

Case 4. Edges $e_l^1, e_l^2, e_m^1, e_m^2$ form a small "diamond" (they pairwise intersect) with $d_l^1 = d_l^2 = d_m^1 = d_m^2$ inside. Assuming that the vertices are labeled in such a way that u_k^j are disjoint, we assign (u_k^j, d_k^j) to e_k^j.

This concludes the argument that $2\,\text{gap}(H) \leq \text{potential}^{\text{int}}(H)$.

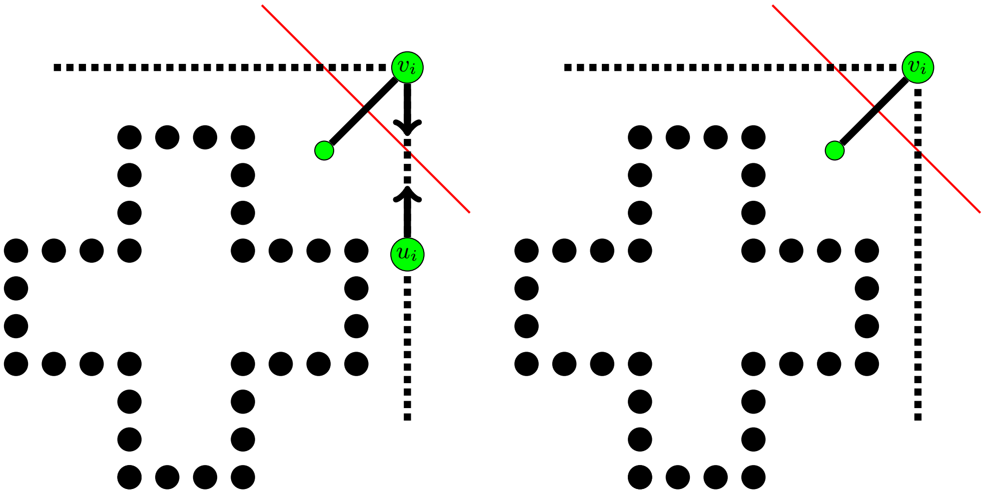

Fig. 10: Two cases appearing in the analysis of the corners. The figure on the left relates to Case I and the figure on the right to Case II.

We will now show that the singular moves $m_{k+1}, \ldots, m_n$ add at least $2 \cdot (n - k) - \text{modifier}(\text{hull}(G))$ to the internal potential of the position (i.e. $\text{potential}(G) - \text{potential}(H) \geq 2 \cdot (n - k) - \text{modifier}(\text{hull}(G))$.

First, observe that there are at most 4 singular moves, that is $n - k \leq 4$. This is because there are two diagonal directions and two sides of a line where the initial cross may be.

Assume that move m_i $(i > k)$ places a singular vertex v_i. Let l_i^1, l_i^2 be half-lines starting from v_i in the direction of the gap line created by move m_i (the dashed lines in Figure 10). There are two possibilities.

Case I. At least one of the half-lines l_i^1, l_i^2 contain a vertex of the position graph. Let u_i denote this vertex. If so, then placing of v_i creates two new edges of internal potential (one starting in v_i in the direction of u_i and another one in u_i in the direction of v_i). The new gap line is compensated.

Case II. Neither of the half-lines l_i^1 and l_i^2 contain a vertex of the position graph. Observe that this is possible only for at most two of the singular moves and each of those moves must create a corner in the hull of G. Moreover, if there are two such moves, the corners are opposite corners of $\text{hull}(G)$.
This concludes the proof.

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