CONDENSATIONS OF PROJECTIVE SETS ONTO COMPACTA

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Abstract. For a coanalytic–complete or $\underset{\sim}{\mathbf{\Pi}}^1_2$–complete subspace X of a Polish space we prove that there exists a continuous bijection of X onto the Hilbert cube $[0,1]^{\mathbb{N}}$. This extends results of Pytkeev. As an application of our main theorem we give an answer to some questions of Arkhangelskii and Christensen.

Under assumption of Projective Determinacy we give also some generalizations of these results to higher projective classes.

1. Introduction.

If X, Y are topological spaces then a continuous bijection $f : X \to Y$ is called a *condensation*. For a topological space X, existence of a compact space K and a condensation $f : X \to K$ is equivalent to existence of a weaker compact topology on X.

Pytkeev proved ([8]) that if X is Borel subspace of a Polish space and X is not σ–compact then there exists a condensation of X onto $[0,1]^{\mathbb{N}}$. Banach in the Scottish Book ([9]) asked a question if there is a condensation of every separable Banach space onto a compact, metrizable space; Kulpa noticed ([5]) that from Pytkeev's theorem follows an affirmative answer to the question. One of the aims of this paper (Theorem 3) is to prove an analog of the theorem of Pytkeev for the classes of coanalytic and $\underset{\sim}{\mathbf{\Pi}}^1_2$–spaces; under additional assumption of Projective Determinacy we prove also a generalization of Pytkeev's result to higher projective classes.

Christensen proved ([3]) that for a given metrizable, σ–compact space X the space $C_p(X) = \{f \in \mathbb{R}^X : f \text{ continuous}\}$ of all continuous functions on X with topology inherited from $\mathbb{R}^X$ is Borel–isomorphic with the Hilbert cube $[0,1]^{\mathbb{N}}$. For the space $\mathcal{N}$ of all irrationals he asked a question if there is a weaker Borel structure on $C_p(\mathcal{N})$ which would be Borel isomorphic with $[0,1]^{\mathbb{N}}$.

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Arkhangelskii proved ([2]) that if X is a σ–compact metrizable space, then $C_p(X)$ has a weaker compact metrizable topology. He formulated a conjecture ([2], p. 1882) that there is no such weaker topology for the space $C_p(\mathcal{N})$. We prove (see **Corollary**) that $C_p(\mathcal{N})$ has a weaker compact metrizable topology; to be more precise, we prove that $C_p(\mathcal{N})$ can be bijectively mapped onto the Hilbert cube $[0,1]^{\mathbb{N}}$. The paper is divided into four parts. After the **Introduction** there is a section where we fix **notation**, define notions and formulate auxiliary facts from descriptive set theory. Next we prove **The Main Theorem** (Theorem 3), which says that every set which is coanalytic–complete or $\underset{\sim}{\mathbf{\Pi}}^1_2$–complete can be mapped onto $[0,1]^{\mathbb{N}}$. Under assumption of Projective Determinacy we extend the statement of the theorem to the case of $\underset{\sim}{\mathbf{\Pi}}^1_n$–complete sets, $n \geq 3$. Theorem 3 is preceded by a lemma on extensions of injective, continuous maps.

In the fourth section, we give a **Corollary**, which is based on results of Andretta and Marcone ([1]) and our main theorem. The **Corollary** contains a solution of the question of Christensen and hypothesis of Arkhangelskii. More precisely, in [1] are shown examples of coanalytic–complete sets and among them there is $C_D(\mathcal{N})$, the projection of the space $C_p(\mathcal{N})$ onto D coordinates, where D is a fixed countable dense subset of $\mathcal{N}$. The image of the projection is a subspace of a separable and metric space $\mathbb{R}^D$. The projection is a bijective, continuous map. Due to our Theorem 3, we infer that there exists a bijective, continuous map from the space $C_D(\mathcal{N})$ onto $[0,1]^{\mathbb{N}}$. The composition of the projection and the condensation of $C_D(\mathcal{N})$ onto $[0,1]^{\mathbb{N}}$ give a condensation of $C_p(\mathcal{N})$ onto $[0,1]^{\mathbb{N}}$. In fact we prove a more general statement: instead of $\mathcal{N}$ one may take an analytic set; under additional assumption of Projective Determinacy, one may take a projective set.

2. Notation and some auxiliary facts.

Non–metrizable spaces appear only in the **Introduction** and in the **Corollary**. In other sections all spaces under consideration are metrizable and separable. Throughout the paper descriptive set theory is applied only to subspaces of Polish spaces.

The symbol I stands for the unit interval $[0,1]$. We say that X is in the class $\underset{\sim}{\mathbf{\Sigma}}^1_1$ or, equivalently, that X is *analytic*, if there exists a continuous function from irrationals $\mathcal{N}$ onto X.

If X is a subspace of a Polish space Y, then we say that X is in the class $\underset{\sim}{\mathbf{\Pi}}^1_1$ or, equivalently, that X is *coanalytic*, if the complement of X in Y is analytic. Continuous images of $\underset{\sim}{\mathbf{\Pi}}^1_1$ sets are called $\underset{\sim}{\mathbf{\Sigma}}^1_2$ sets, their complements are called $\underset{\sim}{\mathbf{\Pi}}^1_2$; for every $n \in \mathbb{N}$ inductively we define in the same way $\underset{\sim}{\mathbf{\Sigma}}^1_n$ and $\underset{\sim}{\mathbf{\Pi}}^1_n$. All these classes are called the *projective classes.*

Whenever we use the symbol Γ, we assume that is one of the class $\underset{\sim}{\mathbf{\Pi}}^1_n$ or $\underset{\sim}{\mathbf{\Sigma}}^1_n$ $(n \geq 1)$. For a space X the symbol $\Gamma(X)$ stands for $\{A \subset X : A \in \Gamma\}$. The projective classes $\underset{\sim}{\mathbf{\Sigma}}^1_n, \underset{\sim}{\mathbf{\Pi}}^1_n$, where $n > 0, n \in \mathbb{N}$ are closed with respect to the closed subsets, countable unions and intersections, homeomorphic images and continuous preimages.

We say that a subset X of a Polish space Y is Γ*–hard* (see [4], def. 22.9) if for every set $C \in \Gamma(2^{\mathbb{N}})$ there exists a continuous map $f : 2^{\mathbb{N}} \to Y$ such that $C = f^{-1}[X]$. We will need more information on the function f; namely that there exists an injection $f : 2^{\mathbb{N}} \to Y$ such that $x \in C$ if and only if $f(x) \in X$. Construction of such an injection from a given continuous mapping is the subject of a theorem of Louveau and Saint–Raymond ([7], also Exercise 26.11 of [4]). Let us notice that the theorem implies

Remark . *(Louveau, Saint–Raymond) If X is Γ–hard subset of a Polish space then X contains closed copies of all Γ subsets of the Cantor set.*

We say that a subset X of a Polish space Y is Γ*–complete* if X is Γ*–hard* in Y and $X \in \Gamma$. We will need the following

Theorem 1. *(Moschovakis, Corollary 39.9 in* [4]*) Let us assume Projective Determinacy and let A belongs to the class $\underset{\sim}{\mathbf{\Sigma}}^1_n$ $n \geq 3, n \in \mathbb{N}$. Then there exists a $\underset{\sim}{\mathbf{\Pi}}^1_n$ subset C of the Cantor set, and a continuous bijection from C onto A.*

Proof. The original result of Moschovakis says (Corollary 39.9, [4]), that under assumption of Projective Determinacy the classes $\underset{\sim}{\mathbf{\Sigma}}^1_{2k+2}$ and $\underset{\sim}{\mathbf{\Pi}}^1_{2k+1}$ $(k \in \mathbb{N})$ have uniformization property. Let A be a subset of a Polish space Y, $A \in \underset{\sim}{\mathbf{\Sigma}}^1_n(Y)$ $(n \geq 3, n \in \mathbb{N})$. There exists $C_0 \in \underset{\sim}{\mathbf{\Pi}}^1_{n-1}(2^{\mathbb{N}})$ and a continuous function $f : C_0 \to Y$ such that $A = f[C_0]$. Let us define a homeomorphism $F : C_0 \to Y \times 2^{\mathbb{N}}$ by the formula $F(x) = (f(x), x)$. The set $G_0 = \{(y, x) \in Y \times 2^{\mathbb{N}} : f(x) = y\} = F[C_0]$ is homeomorphic to C_0, hence belongs to $\underset{\sim}{\mathbf{\Pi}}^1_{n-1}(Y \times 2^{\mathbb{N}})$. According to Moschovakis' theorem, one of the class $\underset{\sim}{\mathbf{\Pi}}^1_{n-1}$, $\underset{\sim}{\mathbf{\Pi}}^1_n$ have uniformization property; in particular, there exists $G_1 \in \underset{\sim}{\mathbf{\Pi}}^1_n(Y \times 2^{\mathbb{N}})$, $G_1 \subset G_0$ such that for all $y \in A$ there exists a unique $x \in 2^{\mathbb{N}}$, such that $(y, x) \in G_1$. We define $C_1 = F^{-1}[G_1] \subset 2^{\mathbb{N}}$. Since it is a homeomorphic image of G_1, the set C_1 belongs to $\underset{\sim}{\mathbf{\Pi}}^1_n$. As we selected exactly one point from every fiber of f, the function f restricted to C_1 is injective and $A = f[C_1]$.$\square$

In the case of $\underset{\sim}{\mathbf{\Sigma}}^1_1$ and $\underset{\sim}{\mathbf{\Sigma}}^1_2$ sets the assumption of Projective Determinacy is not necessary:

Theorem 2. *(Mazurkiewicz, Th. 39.3 in* [6] *for* $\underset{\sim}{\mathbf{\Sigma}}^1_1$*, Kondo, Th. 36.14 in* [4] *for* $\underset{\sim}{\mathbf{\Sigma}}^1_2$*) Let A belong to the class* $\underset{\sim}{\mathbf{\Sigma}}^1_2$*. Then there exists a coanalytic subset C of the Cantor set and a continuous bijection from C onto A.*

3. The Main Theorem.

In this section our reasoning follows Pytkeev's ideas from [8]. Let m be a fixed natural number. We define

$$C_m = \{x \in I^{\mathbb{N}} : \text{there exists } n_0 \text{ such that } x_n = \frac{1}{m+1} \text{ for all } n \geq n_0\}.$$

Let π_k denotes the projection on the k–th ($k \in \mathbb{N}$) axis of the product $I^{\mathbb{N}}$. Let π^1_k, π^2_k ($k \in \mathbb{N}$) denote projections on the appropriate axis of the product $I^{\mathbb{N}} \times I^{\mathbb{N}}$.

Lemma . *Let Γ be one of the class* $\underset{\sim}{\mathbf{\Sigma}}^1_n$, $\underset{\sim}{\mathbf{\Pi}}^1_n$ *($n \geq 1$) and let X be in Γ, $M \subset X$ be a closed subspace, and f be a continuous injective function $f : M \to I^{\mathbb{N}} \times (I^{\mathbb{N}} \setminus C_m)$ such that the image $f[M]$ is in $\Gamma(I^{\mathbb{N}} \times I^{\mathbb{N}})$. Then there exists $g : X \to I^{\mathbb{N}} \times I^{\mathbb{N}}$, a continuous injective function which extends f and such that $g(X \setminus M) \subset I^{\mathbb{N}} \times C_m$. Moreover, the image $g[X]$ is in the family $\Gamma(I^{\mathbb{N}} \times I^{\mathbb{N}})$.*

Proof. (we use a method from [5]) Let h be a homeomorphic embedding of X into $I^{\mathbb{N}}$ such that for every $k \in \mathbb{N}$ the set $\{n \in \mathbb{N} : \pi_n \circ h = \pi_k \circ h\}$ is infinite. Let M_k ($k \in \mathbb{N}$) be a family of closed subsets of X such that $M_k \subset M_{k+1}$ and $\bigcup_{k\in\mathbb{N}} M_k = X \setminus M$. For a number $k \in \mathbb{N}$ we define

$$h^1_k : M_k \cup M \to [0,1]$$
$$h^2_k : M_k \cup M \to [0,1]$$

by the following formulas

$$h^1_k(x) = \begin{cases} \pi^1_k(f(x)) & \text{for } x \in M, \\ \pi_k(h(x)) & \text{for } x \in M_k. \end{cases}$$

$$h^2_k(x) = \begin{cases} \pi^2_k(f(x)) & \text{for } x \in M, \\ \frac{1}{m+1} & \text{for } x \in M_k. \end{cases}$$

Let g^1_k, g^2_k be any continuous functions on X which extend the functions h^1_k, h^2_k ($k \in \mathbb{N}$). We define g^1 and g^2 as diagonal products of g^1_k and g^2_k ($k \in \mathbb{N}$) and the function g as the diagonal product of g_1 and g_2. It is reassured by the definition that the **function g is an extension of** f.

Let k be a natural number. For every $l \in \mathbb{N}$ we fix a number $n_l > k$ such that $\pi_{n_l} \circ h = \pi_l \circ h$. The diagonal product of $\{g^1_{n_l}\}_{l\in\mathbb{N}}$ restricted to M_k is equal to a homeomorphic embedding of M_k into $I^{\mathbb{N}}$. This implies that g (which is the diagonal product of a family of functions which contains all $\{g^1_{n_l}\}_{l\in\mathbb{N}}$) restricted to M_k is also a homeomorphic embedding. Since the class Γ is closed with respect to homeomorphisms and countable unions and that the image $f[M]$ is in Γ, we obtain that the **image** $g[X] = g[\bigcup_{k\in\mathbb{N}} M_k \cup M] = \bigcup_{k\in\mathbb{N}} g[M_k] \cup f[M]$ **is** also in Γ.

Let us check that the **image under** g **of** $X \setminus M$ **is a subset of** $I^{\mathbb{N}} \times C_m$. If $x \in X \setminus M$ then $x \in M_{k_0}$ for some $k_0 \in \mathbb{N}$. For every $k \geq k_0$ ($k \in \mathbb{N}$) we have $h^2_k(x) = \frac{1}{m+1}$. This implies that $g(x) \in I^{\mathbb{N}} \times C_m$.

In the end we should check **injectivity of** g. Let us notice that it follows from:

- the fact that the family $\{M_k\}_{k\in\mathbb{N}}$ is increasing,
- the fact that g restricted to M_k ($k \in \mathbb{N}$), and g restricted to M is injective and
- the fact that $g(M) \cap g(X \setminus M) = \emptyset$. $\square$

For a given space X we say that a family of subsets $X_i \subset X$ $(i \in I)$ is *discrete* if for every $x \in X$ there exists an open set $U \subset X$, $x \in U$ such that U intersects at most one set from the family $\{X_i\}_{i\in I}$. We say that a family of subsets $X_i \subset X$ $(i \in I)$ is *locally finite* if for every $x \in X$ there exists an open set $U \subset X$, $x \in U$ such that $\{i \in I : X_i \cap U \neq \emptyset\}$ is a finite set.

Theorem 3. *If X is $\utilde{\mathbf{\Pi}}^1_1$–complete or $\utilde{\mathbf{\Pi}}^1_2$–complete of a Polish space then there exists a continuous injective map from X onto $I^{\mathbb{N}} \times I^{\mathbb{N}}$. Under Projective Determinacy the same holds for X which is $\utilde{\mathbf{\Pi}}^1_n$–complete ($n \geq 3$) subset of a Polish space.*

Proof of the Theorem. Let $n > 0, n \in \mathbb{N}$ be a fixed natural number and let Y be any $\utilde{\mathbf{\Pi}}^1_n$–complete subset of the Cantor set (one can find examples of such coanalytic sets in Chapter 27 of [4]; universal sets are standard examples of $\utilde{\mathbf{\Pi}}^1_n$–complete sets, $n > 0, n \in \mathbb{N}$). Due to the fact that X is $\utilde{\mathbf{\Pi}}^1_n$–complete and the fact that the family of all $\utilde{\mathbf{\Pi}}^1_n$ subsets of the Cantor set is closed with respect to the multiplication by $\mathbb{N}$ there exists a closed embedding of $i : \mathbb{N} \times Y \to X$. We define $D_k = i[\{k\} \times Y]$ ($k \in \mathbb{N}$), closed subsets of X.

The family $\{D_k\}_{k\in\mathbb{N}}$ is a discrete family of of homeomorphic images of $\utilde{\mathbf{\Pi}}^1_n$–complete subsets of the Cantor set. In particular, according to the **Remark** (see Section 2), for every $k \in \mathbb{N}$ the set D_k contains closed copies of all subsets of the Cantor set which belong to the class $\utilde{\mathbf{\Pi}}^1_n$.

Let $j' : \bigcup_{k\in\mathbb{N}} D_k \to [0,+\infty) \subset \mathbb{R}$ be a function which associates with $x \in D_k$ ($k \in \mathbb{N}$) the natural number k. Let $j : X \to [0,+\infty)$ be a continuous exstension of the function j'. We define $F_k = j^{-1}([k-\frac{1}{2}; k+\frac{1}{2}])$ ($k \in \mathbb{N}$). The family $\{F_k : k \in \mathbb{N}\}$ is a locally finite covering of X consisting of closed subsets of X such that $D_k \subset F_k \setminus \bigcup_{l<k} F_l$ ($k \in \mathbb{N}$).

We define A_k as $\bigcup_{l\le k} F_l$ ($k \in \mathbb{N}$); let $B_k = I^{\mathbb{N}} \times (I^{\mathbb{N}} \setminus \bigcup_{l>k} C_l)$ ($k \in \mathbb{N}$). Let us notice that $\bigcup_{k\in\mathbb{N}} B_k = I^{\mathbb{N}} \times I^{\mathbb{N}}$ and $B_{k+1} = B_k \cup I^{\mathbb{N}} \times C_{k+1}$ ($k \in \mathbb{N}$). We will construct a sequence of functions f_k ($k \in \mathbb{N}$) such that

(1) $f_k : A_k \to I^{\mathbb{N}} \times I^{\mathbb{N}}$,
(2) the image $f_k[A_k]$ is in $\utilde{\mathbf{\Pi}}^1_n(I^{\mathbb{N}} \times I^{\mathbb{N}})$,
(3) f_{k+1} extends f_k,
(4) $B_k \subset f_{k+1}[A_{k+1}] \subset B_{k+1}$ and $f_0[A_0] \subset B_0$.

The union of the functions f_k will be the required bijection of X onto $I^{\mathbb{N}} \times I^{\mathbb{N}}$.

We will start the induction from the definition of the function f_0. Let h be a homeomorphic embedding of X into $I^{\mathbb{N}}$ and let $f_0 : A_0 \to I^{\mathbb{N}} \times I^{\mathbb{N}}$ be the diagonal product of the restriction of h to A_0 and the constant function equal $(1,1,\ldots) \in I^{\mathbb{N}}$. Since the homeomorphic image $f_0[A_0]$ is in $\utilde{\mathbf{\Pi}}^1_n(I^{\mathbb{N}} \times I^{\mathbb{N}})$ and $f_0[A_0] \subset B_0$, the function f_0 fulfills all the inductive requirements.

Let k be a natural number and let us assume that we constructed f_k with the properties 1–4. Since $f_k[A_k]$ is in $\utilde{\mathbf{\Pi}}^1_n(I^{\mathbb{N}} \times I^{\mathbb{N}})$ and B_k is a Borel set, the difference $B_k \setminus f_k[A_k]$ is in $\utilde{\mathbf{\Sigma}}^1_n(I^{\mathbb{N}} \times I^{\mathbb{N}})$.

According to the Mazurkiewicz–Kondo theorem (for $n = 1, 2$) or Moschovakis theorem (for $n \ge 3$), there exists a subspace C of the Cantor set which is in the class $\utilde{\mathbf{\Pi}}^1_n(2^{\mathbb{N}})$ and an injective continuous mapping g from C onto $B_k \setminus f_k[A_k]$. Since F_{k+1} contains as a closed subset a $\utilde{\mathbf{\Pi}}^1_n$–complete set D_{k+1}, due to the **Remark** (see Section 2) we may assume that C is a closed subset of F_{k+1}.

Let us define $f'_k : C \cup A_k \to I^{\mathbb{N}} \times I^{\mathbb{N}}$ as the common extension of f_k and g. Since the image $f'_k[C \cup A_k]$ is equal to B_k, it follows that $f'_k[C \cup A_k]$ is Borel and $f'_k[C \cup A_k]$ is a subset of $I^{\mathbb{N}} \times (I^{\mathbb{N}} \setminus C_{k+1})$. We apply the Lemma for $m = k+1$ to the function f'_k and define $f_{k+1} : A_{k+1} \to I^{\mathbb{N}} \times I^{\mathbb{N}}$ as the extension of f'_k given by the Lemma.

Only condition 4. may need some explanation. Since the image of the function f'_k is equal to B_k, the image of f_{k+1} contains B_k as well. This proves that the first inclusion holds. Due to the Lemma the image of the extension is contained in the union of the set $I^{\mathbb{N}} \times C_{k+1}$ and the image of $f'_k[C \cup A_k] = B_k$. According to the definition of the family $\{B_k\}_{k\in\mathbb{N}}$ the union is equal to B_{k+1}. This proves that the second inclusion holds. $\square$

4. Corollary.

Let A be a projective subspace of a Polish space and let us assume that A is not σ–compact. For analytic A we set $n=1$ and otherwise $n\in\mathbb{N}$ is the natural number such that $A\in\utilde{\mathbf{\Sigma}}^1_n$ but $A\notin\utilde{\mathbf{\Sigma}}^1_{n-1}$.

Let us fix a metric d on A. The symbol $C_p(A)$ stands for the space of all continuous real–valued functions on A with the topology of pointwise convergence. **We prove that $C_p(A)$ can be condensed onto $I^{\mathbb{N}}$; we assume Projective Determinacy unless A is analytic**. Let D be a fixed dense countable subset of A. The symbol π_D stands for the projection of $\mathbb{R}^{\mathcal{N}}$ onto $\mathbb{R}^D$. We define

$$\begin{aligned} C_D(A) \quad &= \{f\in\mathbb{R}^D : f \text{ is a restriction of a function from } C_p(A)\} = \\ &= \pi_D[C_p(A)] \end{aligned}$$

It was proved by Andretta and Marcone (Lemma 2.3 of [1]), that if A is $\utilde{\mathbf{\Sigma}}^1_n$ then $C_D(A)$ is a $\utilde{\mathbf{\Pi}}^1_n$ subset of $\mathbb{R}^D$. Let us notice that $\pi_D|C_p(A)$ is a condensation of the space $C_p(A)$ onto the space $C_D(A)$.

It is enough to check that $C_D(A)$ fulfills the assumptions of Theorem 3. It follows from results of Andretta and Marcone, who proved that the set $C_D(A)$ is $\utilde{\mathbf{\Pi}}^1_n$–complete subset of the Polish space $\mathbb{R}^D$ ([1], Theorem 3.3 for analytic A, Theorem 4.3 for projective A).

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References

[1] A. Andretta, A. Marcone, *Pointwise convergence and the Wadge hierarchy*, Comment. Math. Univ. Carolinae **41** (2001), 159–172.

[2] A.V. Arkhangelskii, *On condensations of C_p–spaces onto compacta*, Proc. Amer. Math. Soc. **128** (2000), 1881–1883.

[3] J.P.R. Christensen, *Topology and Borel Structure, North–Holand*, Amsterdam, **1974**.

[4] A.S. Kechris, *Classical Descriptive Set Theory*, Springer–Verlag, New York, 1994.

[5] W. Kulpa, *On a problem of Banach*, Colloquium Math. **56** (1988), 255–261.

[6] K. Kuratowski, *Topology*, vol. I, New York, 1966.

[7] A. Louveau, J. Saint–Raymond, *Borel classes and closed games: Wadge–type results and Hurewicz–type results*, Trans. Amer. Math. Soc. **304** (1987), 431–467.

[8] E.G. Pytkeev, *The upper bounds of topologies (in Russian)*, Mat. Zametki **20** no. 4 (1976), 489–500.

[9] *The Scottish Book*, ed. R.D. Mauldin, Birkhauser, Boston, **1981**.

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