Distribution theory

Aim: find a class of objects that

(0) is useful for PDE

(1) contains all continuous functions

(2) objects posess partial derivatives (in the same class) that coincides, for differentiable functions, with the classical one

(3) standard differentiation rules should hold

(4) proper algebraic/analytic structure in the class (Banach space?)

Test functions

For any $K \Subset \mathbb{R}^n$ we set $\mathcal{D}_K = \{ f \in C^\infty(\mathbb{R}^n) : \operatorname{supp} f \subset K \}$

Then, for $\Omega \subset \mathbb{R}^n$, $\mathcal{D}(\Omega) = \bigcup_{K \Subset \Omega} \mathcal{D}_K$.

Topology on $\mathcal{D}_K$

Set $\|f\|_N = \sup\{|D^\alpha f(x)| : |\alpha| \leq N, x \in \Omega\}$

$$\left\{\begin{array}{l} \alpha = (\alpha_1, \dots, \alpha_n) \text{ multiindex } \in (\mathbb{N} \cup \{0\})^n \\ |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n \\ D^\alpha = \dfrac{\partial^n}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}} \end{array}\right.$$

for $f \in \mathcal{D}_K$ $\|\cdot\|_N$ is a norm

- it satisfies positive homogeneity & triangle ineq.

~~but it might be zero for~~

However, for a fixed N, it is no more than $C^N(K)$ - norm, it does not capture at all the behaviour of high-order derivatives.

Since $D_K \subset C^\infty$, we would like to have

$\varphi_n \xrightarrow[\mathcal{D}_K]{} \varphi$ iff $\forall \alpha$ $D^\alpha \varphi_n \rightrightarrows D^\alpha \varphi$ (convergence in C^∞)

thus we have on $\mathcal{D}_K$ a topology given by a countable family of norms $\|\cdot\|_N$. } Fréchet space

Equivalently, $d(\varphi, \psi) = \sum_{N=0}^{\infty} \dfrac{\|\varphi - \psi\|_N}{1 + \|\varphi - \psi\|_N} \cdot 2^{-N}$ a metric

Fact: $\mathcal{D}_K$ is complete with this metric.

<u>Topology on $\mathcal{D}(\Omega)$</u>

We can find $K_1 \subset K_2 \subset \dots$ compact subsets of Ω s.t. $\Omega = \bigcup_i K_i$

~~On $\mathcal{D}(\Omega)$~~ Clearly, $\mathcal{D}_K \hookrightarrow \mathcal{D}(\Omega)$.

We choose the strongest topology such that the inclusions $\mathcal{D}_{K_i} \hookrightarrow \mathcal{D}(\Omega)$ are continuous (inductive topology).

More transparently: a function on $\mathcal{D}(\Omega)$ is continuous, if its ~~is~~ restriction to $\mathcal{D}_{K_i}$, for any K_i, is continuous in the topology of $\mathcal{D}_{K_i}$

In the language of convergent sequences:
A sequence (φ_m) in $\mathcal{D}(\Omega)$ is convergent iff there exists $K \subset\subset \Omega$ s.t. $\forall_m$ supp $\varphi_m \subset K$ and (φ_m) is convergent in $\mathcal{D}_K$: (to some φ)

$$\forall_{N \in \mathbb{N} \cup \{0\}} \quad \|\varphi_m - \varphi\|_N \xrightarrow[m \to \infty]{} 0$$

<u>Wait:</u> We have a countable number of norms in each $\mathcal{D}_{K_i}$, clearly $\mathcal{D}(\Omega) = \bigcup_i \mathcal{D}_{K_i}$ ← we have a countable family of (semi)norms $\|f\|_{N,K_i} = \cancel{\sup}$

$$= \sup\{|D^\alpha f(x)| : |\alpha| \le N, x \in K_i\}.$$

Why not just take
$$d(f,g) = \sum_{i,j} \frac{\|f-g\|_{i,K_j}}{1+\|f-g\|_{i,K_j}} 2^{-(i+j)} \quad ?$$

Exercise: Prove that with the above metric $\mathcal{D}(\Omega)$ is not complete.

Fact: The topology we described on $\mathcal{D}(\Omega)$ is not metrizable, however, it is complete in the sense that if (φ_n) is a Cauchy sequence in $\mathcal{D}(\Omega)$

(for every nbhd ~~of~~ $\mathcal{U}$ of 0 there exists n_0 s.t. $\forall_{n,m>n_0}$ $\varphi_n - \varphi_m \in \mathcal{U}$)

then φ_n is convergent.

Precise description of topology of $\mathcal{D}(\Omega)$ (~~what sets~~ (open sets etc.) in Rudin's F.A.

Distributions in Ω are continuous linear functionals on $\mathcal{D}(\Omega)$; their space is denoted by $\mathcal{D}'(\Omega)$.

Value of a distribution S on test function φ will be denoted by $\langle S, \varphi \rangle$.

Topology on $\mathcal{D}'(\Omega)$

A sequence (S_k) in $\mathcal{D}'(\Omega)$ converges to $S \in \mathcal{D}'(\Omega)$ if for every $\varphi \in \mathcal{D}(\Omega)$

$$\langle S_k, \varphi \rangle \longrightarrow \langle S, \varphi \rangle$$

(this is known as $*$-weak topology).

Theorem A linear functional S on $\mathcal{D}(\Omega)$ is a distribution (i.e. $S \in \mathcal{D}'(\Omega)$) if and only if for any $K \subset\subset \Omega$ there exists $N \in \mathbb{N} \cup \{0\}$ and $C > 0$ such that

$$\left.\begin{array}{l} |\langle S, \varphi \rangle| \leq C \, \|\varphi\|_N \\ \text{for all } \varphi \in \mathcal{D}_K. \end{array}\right\} (*)$$

This is just rephrasing the fact that the topology on $\mathcal{D}(\Omega)$ is such that S is continuous on $\mathcal{D}(\Omega)$ iff $S|_{\mathcal{D}_K}$ is continuous on $\mathcal{D}_K$. Continuity on $\mathcal{D}_K$ means precisely $(*)$.

Def: If we can find N such that $(*)$ holds for all $K \subset\subset \Omega$ (N independent of K) then the smallest such N is the <u>order</u> of S.

Examples

1. Every $f \in L^1_{loc}(\Omega)$ ← functions that are integrable on all compact subsets of Ω

defines a distribution T_f

$$\langle T_f, \varphi \rangle = \int_{\Omega} f \varphi$$

$\Big\}$ φ has compact support, so this makes sense.

~~Exercise:~~ $\forall_{K \subset\subset \Omega}$ $|\langle T_f, \varphi\rangle| \leq \int_K |f| \cdot \|\varphi\|_0$

order is 0.

2. For any $x \in \Omega$ we have the Dirac distribution δ_x

$$\langle \delta_x , \varphi \rangle = \varphi(x).$$

Since $\forall_{\varphi \in \mathcal{D}(\Omega)} \forall_{K \subset\subset \Omega}$ $|\langle \delta_x, \varphi\rangle| \leq \|\varphi\|_0$,

this is a distribution of order 0.

3. Similarly, every Radon measure μ

- $\mu(A) = \sup_K \{\mu(K) : K \subset\subset A\}$
- every pt has a nbhd with finite measure

defines a distribution

$$\langle T_\mu, \varphi \rangle = \int_\Omega \varphi \, d\mu$$

(of order 0).

4. $\langle \delta_x', \varphi \rangle = -\varphi'(x)$ is a distribution of order 1. $(\Omega \subset \mathbb{R})$

5. Fix a sequence (x_i) in Ω that has no accumulation points in Ω.

Then $T = \sum_{i=1}^{\infty} \delta_{x_i}^{(i)}$; $\langle T, \varphi \rangle = \sum_{i=1}^{\infty} \varphi^{(i)}(x_i)$

is a distribution of infinite order.

Operations on distributions

Multiplication by smooth functions

For any smooth $f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$)

$$\langle f \cdot T, \varphi \rangle := \langle T, f\varphi \rangle$$

(for $T = T_g$, with $g \in L^1_{loc}$,

$$\langle f \cdot T_g, \varphi \rangle = \langle T_g, f\varphi \rangle = \int g f \varphi = \langle T_{fg}, \varphi \rangle)$$

Problem: Solve the equation $x \cdot T = 0$

(find all such $T \in \cancel{\mathcal{D}'(\mathbb{R}^3)}\ \mathcal{D}'(\mathbb{R})$)

Lemma (preparation lemma)

For any $\varphi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{D}(\mathbb{R})$ s.t. $\psi(0) = 1$ there exists $\tilde{\varphi} \in \mathcal{D}(\mathbb{R})$ s.t. $\forall_x$

$$\varphi(x) = \varphi(0)\psi(x) + x\tilde{\varphi}(x)$$

Proof:

Let $\alpha(x) = \varphi(x) - \varphi(0)\psi(x)$.

Then, by Taylor's formula with integral remainder

$$\alpha(x) = \underset{\overset{\|}{0}}{\alpha(0)} + x\alpha'(0) + \int_0^x \frac{\alpha''(t)}{2}(x-t)\,dt$$

Change of variables: $t = xs$ $\quad s = \frac{t}{x}$
$dt = x\,ds$

$$\alpha(x) = x\alpha'(0) + \int_0^1 \frac{\alpha''(xs)}{2} x(1-s)\,x\,ds$$
$$= x\underbrace{\left[\alpha'(0) + \frac{x}{2}\int_0^1 \alpha''(xs)(1-s)\,ds\right]}_{\tilde{\varphi}(x)}$$

Obviously, $\tilde{\varphi}$ is smooth, and satisfies
$$\varphi(x) = \varphi(0)\psi(x) + x\tilde{\varphi}(x),$$
since φ and ψ have compact supports, so has $\tilde{\varphi}$.

Solution

Fix $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi(0) = 1$.
Let $C = \langle T, \psi\rangle$.
Now, for any $\varphi \in \mathcal{D}(\mathbb{R})$,
$$\langle T, \varphi\rangle = \langle T, \varphi(0)\psi + x\tilde{\varphi}\rangle =$$
$$= \varphi(0)\langle T, \psi\rangle + \langle \underset{\substack{\| \\ 0}}{xT}, \tilde{\varphi}\rangle$$
$$= C\varphi(0) = \langle C\delta_0, \varphi\rangle$$

Thus $T = C\delta_0$.
Clearly, for any $C \in \mathbb{R}\ (\mathbb{C})$ $T = C\delta$ satisfies $xT = 0$.

<u>Problem</u>: Find all $T \in \mathcal{D}'(\mathbb{R})$ such that $x^2 T = 0$.

<u>Question</u> (L. Schwartz, 1951)

<u>Division problem for distributions</u>

Given $T \in \mathcal{D}'(\mathbb{R}^n)$ and polynomial P, ~~Do~~ does there exist $S \in \mathcal{D}'(\mathbb{R}^n)$ such that $T = P \cdot S$?

<u>Answers</u>: S. Łojasiewicz (1958)

- Yes, not only for polynomial P, but for any analytic P.

L. Hörmander (1958, as well)

- yes, for polynomial P.

Differentiation of distributions

For any multiindex α,

$$\langle D^\alpha T, \varphi \rangle := (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle$$

(for smooth g, $\langle D^\alpha T_g, \varphi \rangle = (-1)^{|\alpha|} \langle T_g, D^\alpha \varphi \rangle$

$$= (-1)^{|\alpha|} \int g \, D^\alpha \varphi \underset{\substack{\uparrow \\ \text{int. by parts}}}{=} (-1)^{|\alpha|} \cdot (-1)^{|\alpha|} \int D^\alpha g \cdot \varphi$$

$$= \langle T_{D^\alpha g}, \varphi \rangle \Big)$$

Exercise: Prove that

$$D^i (f \cdot T) = D^i f \cdot T + f \cdot D^i T$$

~~for any~~ (D^i - derivative in direction of x_i)

Shifts: Suppose $\Omega = \mathbb{R}^n$.

We have $\tau_h : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$

$$\tau_h f(x) = f(x-h).$$

Then, for $T \in \mathcal{D}'(\mathbb{R}^n)$,

$$\langle \tau_h T, \varphi \rangle := \langle T, \tau_{-h} \varphi \rangle$$

Check that $\tau_h T_g = T_{\tau_h g}$

Convolutions

Denote $\check{u}(y) = u(-y)$, then for any functions u, v

$$u * v\,(x) = \int_{\mathbb{R}^n} u(y)\, v(x-y)\, dy = \int_{\mathbb{R}^n} u \cdot \tau_x \check{v}$$

By analogue, for $T \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$(T * \varphi)(x) := \langle T, \tau_x \check{\varphi} \rangle = \langle \tau_{-x} T, \check{\varphi} \rangle$$

Exercises

For any $T \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$

(1) $T * \varphi \in C^\infty(\mathbb{R}^n)$

(2) $\forall_\alpha \; D^\alpha (T * \varphi) = (D^\alpha T) * \varphi = T * D^\alpha \varphi$

(3) $\tau_h (T * \varphi) = (\tau_h T) * \varphi = T * \tau_h \varphi$

(4) for any $\psi \in \mathcal{D}(\mathbb{R}^n)$

$$T * (\varphi * \psi) = (T * \varphi) * \psi$$

<u>Theorem</u>: The set $\{T_f : f \in C^\infty(\mathbb{R}^n)\}$ is dense in $\mathcal{D}'(\mathbb{R}^n)$.

<u>Proof</u>:

Take $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\varphi \geqslant 0$, $\int_{\mathbb{R}^n} \varphi = 1$

and let $\varphi_m(x) = m^n \varphi(mx)$ (approximate unit)

<u>Fact</u>: for any $f \in C(\mathbb{R}^n)$ $\quad f * \varphi_n \rightrightarrows f$ on any compact subset of $\mathbb{R}^n$.

Choose any $\chi \in \mathcal{D}(\mathbb{R}^n)$, then there exists $K \subset\subset \mathbb{R}^n$ such that the supports of all $\chi * \varphi_m$ are contained in K (e.g. $K = \operatorname{supp}\varphi \underset{\substack{\uparrow \\ \text{Minkowski sum}}}{+} \operatorname{supp}\chi$).

Likewise, ~~thes~~ for any α, the supports of $D^\alpha(\chi * \varphi_m) = D^\alpha \chi * \varphi_m$ are in K.

Since $\forall_\alpha \; D^\alpha(\chi * \varphi_m) = D^\alpha \chi * \varphi_m \underset{\substack{\text{on } K, \\ \text{thus on } \mathbb{R}^n}}{\rightrightarrows} D^\alpha \chi$

we have $\chi * \varphi_m \longrightarrow \chi$ in $\mathcal{D}(\mathbb{R}^n)$.

Take any $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$.

$$\langle T, \psi \rangle = \langle T, (\check{\psi})^{\vee} \rangle = (T * \check{\psi})(0)$$

$$= \lim_{m \to \infty} (T * \check{\psi}) * \varphi_m (0)$$

$$= \lim_{m \to \infty} (T * \varphi_m) * \check{\psi} (0)$$

$$= \lim_{m \to \infty} \langle T_{T * \varphi_m}, \psi \rangle$$

thus $T_{\underbrace{T * \varphi_m}_{\in C^\infty}} \longrightarrow T$ in $\mathcal{D}'(\mathbb{R}^n)$.

Exercises

1. Prove that if $\Lambda : \mathcal{D}(\Omega) \to C^\infty(\Omega)$ is a continuous linear map such that for any α $D^\alpha \Lambda = \Lambda D^\alpha$ then $\exists_{T \in \mathcal{D}'(\Omega)}$ s.t. $\forall_{\varphi \in \mathcal{D}(\Omega)}$ $\Lambda(\varphi) = T * \varphi$.

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2. We say that $T \in \mathcal{D}'(\Omega)$ vanishes on $V \overset{\text{open}}{\subset} \Omega$ if $\langle T, \varphi \rangle = 0$ for all φ with $\operatorname{supp} \varphi \subset V$.
supp T is the complement of the maximal subset of Ω on which T vanishes:

$$\operatorname{supp} T = \bigcap \{ \Omega \setminus V : T \text{ vanishes on } V \}$$

$\operatorname{supp} T_f = \operatorname{supp} f$; $\operatorname{supp} \delta_x = \{x\}$.

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2. Suppose T has compact support. Show that for any $\varphi \in \mathcal{D}$ $T * \varphi$ has compact support.

Using this result we can define $S * T$ for any $S \in \mathcal{D}'(\Omega)$ and $T \in \mathcal{D}'(\Omega)$ with compact support:

$$\langle S * T, \varphi \rangle = \langle S, T * \varphi \rangle.$$

3. $S * \delta = S$