Vitali's covering lemma

A Vitali covering of a set $E \subset \mathbb{R}^n$ is a family $\mathcal{B}$ of balls in $\mathbb{R}^n$ s.t

(•) $\bigcup \mathcal{B} \supset E$ (i.e. $\mathcal{B}$ is a covering)

(•) $\inf\{\operatorname{diam}(B) : B \in \mathcal{B} \wedge x \in B\} = 0$

(Vitali's condition)

Lemma (Vitali)

Let E be a subset of $\mathbb{R}^n$, with finite outer measure. Then there exists a countable subfamily $\{B_i\}_{i \in \mathbb{N}}$ ~~st~~. of $\mathcal{B}$ such that B_i are disjoint,

$$|E \setminus \bigcup_i B_i| = 0, \quad |\bigcup_i B_i| < \infty.$$

Proof of Vitali's lemma

Step 1 Simplifications

By possibly ~~thrw~~ throwing away some balls from $\mathcal{B}$ we can assume, without loss of generality, that

- $\forall_{B \in \mathcal{B}}$ diam $B \leq 1$ (throw away big ball's, Vitali's condition ensures we still have a covering)

- there exists an open $\mathcal{U}$ s.t. $\bigcup \mathcal{B} \subset \mathcal{U}$, $|\mathcal{U}| < \infty$.

  (We do this by choosing a subfamily $\tilde{\mathcal{B}}$ out of $\mathcal{B}$: Fix $\underset{\text{open}}{\mathcal{U} \supset E}$ s.t. $|\mathcal{U}| < \infty$,
  $$\tilde{\mathcal{B}} = \bigcup_{x \in E} \{ B \in \mathcal{B} : x \in B, \text{ diam } B < d(x, \mathbb{R}^n \setminus \mathcal{U}) \})$$

- All balls in $\mathcal{B}$ are closed.

  (Suppose not all balls are closed, then we consider $\tilde{\mathcal{B}} = \{ \overline{B} : B \in \mathcal{B} \}$, prove the lemma for $\tilde{\mathcal{B}}$ and notice that $|\bigcup_i \partial B_i| = 0$, thus the lemma holds for $\{B_i\}$ as well.)

Step 2. Inductive construction of $\{B_i\}$

For B_1, choose any $B \in \mathcal{B}$ such that

$$\operatorname{diam} B_1 > \frac{1}{2} \sup_{B \in \mathcal{B}} \operatorname{diam} B.$$

Denote by $\mathcal{B}_1$ the family of all balls in $\mathcal{B}$ disjoint with B_1:

$$\mathcal{B}_1 = \{ B \in \mathcal{B} : B \cap B_1 = \emptyset \}.$$

inductively, for B_k choose any ball in $\mathcal{B}_{k-1}$ such that

$$\operatorname{diam} B_k > \frac{1}{2} \sup_{B \in \mathcal{B}_{k-1}} \operatorname{diam} B.$$

and set $\mathcal{B}_k = \{ B \in \mathcal{B}_{k-1} : B \cap B_k = \emptyset \}$.

Step 3

Note that $\displaystyle\sum_{k=1}^{\infty} |B_k| \le |\mathcal{U}| < \infty$,

thus $|B_k| \approx (\operatorname{diam} B_k)^n \xrightarrow[k\to\infty]{} 0$,

therefore $\displaystyle\sup_{B \in \mathcal{B}_k} \operatorname{diam} B \xrightarrow[k\to\infty]{} 0$.

Step 4

Fix $N \in \mathbb{N}$ and let $x \in E \setminus \bigcup_{i=1}^{N} B_i$

Choose $B_0 \in \mathcal{B}$ such that

- $x \in B_0$
- B_0 is disjoint with $B_1, B_2, \dots, B_N$. $\Big| \Rightarrow B_0 \in \mathcal{B}_N$

(a ball like that exists, because $d(x, \bigcup_{i=1}^{N} B_i)$ is positive + Vitali's condition).

Can it happen that $\forall_{k \in \mathbb{N}} \; B_0 \cap B_k = \emptyset$?

No, because $\sup_{B \in \mathcal{B}_k} \operatorname{diam} B \xrightarrow[k \to \infty]{} 0$, thus

B_0 does not qualify to be in all $\mathcal{B}_k$.

Let k_0 be the smallest number such that

$B_{k_0} \cap B_0 \neq \emptyset$. Then $k_0 > N$,

B_0 and B_{k_0} are in ~~$\mathcal{B}_{k_0-1}$~~ $\mathcal{B}_{k_0-1}$

$$\operatorname{diam} B_0 \leq \sup_{B \in \mathcal{B}_{k_0-1}} \operatorname{diam} B < 2 \operatorname{diam} B_{k_0}$$

$$\left(\operatorname{diam} B_0 < 2 \operatorname{diam} B_{k_0}\right) + \left(B_0 \cap B_{k_0} \neq \emptyset\right)$$

$$\Downarrow$$

$$B_0 \subset 5 B_{k_0}$$

Step 5

$$E \setminus \bigcup_{k=1}^{\infty} B_k \subset E \setminus \bigcup_{k=1}^{N} B_k \subset \bigcup_{i>N} 5B_i$$

but this can be made arbitrarily small (in measure) by increasing N:

$$\left|E \setminus \bigcup_{k=1}^{\infty} B_k\right| \leq \left|\bigcup_{i>N} 5B_i\right| \leq 5^n \sum_{i>N} |B_i| \xrightarrow{N\to\infty} 0$$

This completes the proof. □

Finite version of Vitali's lemma

For any $\varepsilon > 0$ we can choose, out of a Vitali's covering $\mathcal{B}$ of E, a finite family $\{B_1, \dots, B_N\}$ such that B_i are disjoint, $\left|\bigcup_{i=1}^{N} B_i\right| < \infty$ and $\left|E \setminus \bigcup_{i=1}^{N} B_i\right| < \varepsilon$.

Proof: Take $B_1, \dots, B_N$ from the infinite version, for N suff. large.

<u>Application</u>

<u>Theorem</u>: A monotonous $f: \cancel{[a,b]}\ \mathbb{R} \to \mathbb{R}$ is differentiable a.e. and $\int_a^b |f'(x)|\,dx \le f(b)-f(a)$ for all $a<b$.

<u>Remark</u>: for f non-decreasing $f'(x) \ge 0$ for all x s.t. $f'(x)$ exists, thus we have

$$\int_a^b f'(x)\,dx \le f(b)-f(a).$$

Clearly, it is enough to prove the theorem for non-decreasing f.

<u>Proof</u>: For any $x \in [a,b]$ set

$$\left.\begin{aligned} D^{\pm} f(x) &= \limsup_{h \to 0^{\pm}} \frac{f(x+h)-f(x)}{h} \\ D_{\pm} f(x) &= \liminf_{h \to 0^{\pm}} \frac{f(x+h)-f(x)}{h} \end{aligned}\right\} \text{Dini's derivatives}$$

Obviously $D^+ f(x) \ge D_+ f(x)$ and $D^- f(x) \ge D_- f(x)$.

We shall prove that $D_+ f(x) \ge D^- f(x)$

and $D_- f(x) \ge D^+ f(x)$

for a.e. $x \in [a,b]$, which is enough to prove differentiability a.e.

Set

$$A^N_{\alpha,\beta} = \{x \in (-N,N) : D_+f(x) < \beta < \alpha < D^-f(x)\}$$

for $N > 0$, $\alpha > \beta$.

We fix $N, \alpha, \beta, \varepsilon$ and approximate $A^N_{\alpha,\beta}$ by an open $\mathcal{U}$:

$$A^N_{\alpha,\beta} \subset \mathcal{U}, \quad |\mathcal{U}| < |A^N_{\alpha,\beta}| + \varepsilon.$$

The set of all intervals $[a,b] \subset \mathcal{U}$ such that $f(b) - f(a) \overset{(*)}{<} \beta(b-a)$ is a Vitali covering of $A^N_{\alpha,\beta}$: for any $x \in A^N_{\alpha,\beta}$

$$\liminf_{h \to 0^+} \frac{f(x+h) - f(x)}{h} < \beta$$

$\Longleftrightarrow$ there exist arbitrarily short intervals $[\underset{a}{\overset{\shortparallel}{x}}, \underset{b}{\overset{\shortparallel}{x+h}}]$ such that $(*)$ is satisfied.

By Vitali's lemma, we choose a finite number of disjoint intervals $[a_i, b_i]$ $i = 1, \dots, K$ s.t. $|A_{\alpha,\beta} \setminus \bigcup_{i=1}^{K} [a_i, b_i]| < \varepsilon$; $f(b_i) - f(a_i) < \beta(b_i - a_i)$

Thrm

f non-decreasing $\Rightarrow$
$\Rightarrow \alpha, \beta > 0$.

Then

$$\sum_{i=1}^{K}[f(b_i)-f(a_i)] < \beta \sum_{i=1}^{K}(b_i-a_i)$$

$$< \beta\left[\mu^*(A^N_{\alpha,\beta}) + \varepsilon\right]$$

Now we look into each of the intervals $[a_i, b_i] =: I_i$

Inside I_i, the intervals $J = [c,d]$ such that $f(d)-f(c) > \alpha(d-c)$ form a Vitali covering of $A^N_{\alpha,\beta} \cap I_i$.

We can choose out of them a finite subfamily $J_1, \dots, J_{k_i}$ such that

$$\mu^*\left(A^N_{\alpha,\beta} \cap I_i \setminus \bigcup_{j=1}^{k_i} J_j\right) < \frac{\varepsilon}{2^i}$$

If $J_j = [c_j, d_j]$, then, by monotonicity of f,

$$f(b_i) - f(a_i) \geq \sum_{j=1}^{k_i} \cancel{f(d_j)} \left(f(d_j) - f(c_j)\right)$$

thus

$$\left(\mu^*(A^N_{\alpha,\beta}) + \varepsilon\right)\beta > \sum_{i=1}^{K}\underbrace{[f(b_i)-f(a_i)]}_{\overset{f}{\Delta} I_i} \geq \sum_{i=1}^{K}\sum_{j=1}^{k_i}[f(d_j)-f(c_j)]$$

$$\cancel{\sum_{i=1}^{K} \overset{f}{\Delta} I_i \geq \sum_{i=1}^{K}\sum_{j}} > \alpha \sum\sum (d_j - c_j)$$

$$\geq \alpha \sum_{i=1}^{k} \left[\mu^*(A^N_{\alpha,\beta} \cap I_i) - \frac{\varepsilon}{2^i} \right]$$

$$> \alpha \left(\mu^*(A^N_{\alpha,\beta}) - 2\varepsilon \right)$$

Since ε is arbitrary and $\alpha > \beta$, this implies $\mu^*(A^N_{\alpha,\beta}) = 0$.

Summing over all natural N and rational α, β we get that the set where

$$D_+ f(x) < D^- f(x)$$

is of measure 0.

The other inequality is proved analogously.

What remains to prove is that

1. f' (that, as we proved, exists a.e) is measurable

2. $\forall_{a<b} \int_a^b f'(x)dx \leq f(b) - f(a)$.

1. Let $\tilde{f}(x) = \begin{cases} f(x) & x \in [a,b] \\ f(b) & x > b \end{cases}$

Then $f'(\cdot)$ is a pointwise limit of

$$g_n(\cdot) = n\left(\tilde{f}\left(\cdot + \tfrac{1}{n}\right) - \tilde{f}(\cdot)\right). \geq 0.$$

2. Fatou's lemma

$$\int_a^b f'(x)\,dx \overset{\downarrow}{\leq} \liminf_{n\to\infty} \int_a^b g_n(x)\,dx$$

$$= \liminf_{n\to\infty}\left[n\int_a^b \tilde{f}\left(x+\tfrac{1}{n}\right)dx - n\int_a^b \tilde{f}(x)\,dx\right] =$$

$$= \liminf_{n\to\infty}\left[n\int_a^{b-\frac{1}{n}} \tilde{f}\left(x+\tfrac{1}{n}\right)dx + n\int_{b-\frac{1}{n}}^{b} \tilde{f}\left(x+\tfrac{1}{n}\right)dx - n\int_a^b f(x)\,dx\right]$$

$$= \liminf_{n\to\infty}\left[n\int_{a+\frac{1}{n}}^{b} f(x)\,dx - n\int_a^b f(x)\,dx + n\int_{b-\frac{1}{n}}^{b} f(b)\,dx\right]$$

$$= \liminf_{n\to\infty}\left[-n\int_a^{a+\frac{1}{n}} f(x)\,dx + f(b)\right] \leq f(b) - f(a)$$

here $f(x) \geq f(a)$

Functions of bounded variation

For any partition ν defined by

$$a = x_0 < x_1 < \dots < x_k = b$$

of an interval $[a,b]$

we write
$$p_\nu(f) = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+$$
$$n_\nu(f) = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^-$$
$$t_\nu(f) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| = n_\nu + p_\nu$$

Clearly, $p_\nu - n_\nu = f(b) - f(a)$ for any ν.

Taking suprema over all partitions of $[a,b]$ we get

$\cancel{P_{[a,b]}}$

$$P_a^b(f) = \sup_\nu p_\nu(f)$$ positive variation

$$N_a^b(f) = \sup_\nu n_\nu(f)$$ negative variation

$$T_a^b(f) = \sup_\nu t_\nu(f)$$ total variation over interval $[a,b]$

The relations

$$T_a^b(f) = P_a^b(f) + N_a^b(f), \quad f(b) - f(a) = P_a^b(f) - N_a^b(f)$$

hold.

Proof:

$$p_\nu(f) = n_\nu(f) + f(b) - f(a) \quad \Big/ \sup_\nu$$

$$\Rightarrow P_a^b(f) = N_a^b(f) + f(b) - f(a) \, ;$$

$$t_\nu(f) = p_\nu(f) + n_\nu(f) =$$

$$= 2p_\nu(f) - (f(b) - f(a)) \quad \Big/ \sup_\nu$$

$$T_a^b(f) = 2P_a^b(f) - (f(b) - f(a))$$

$$= P_a^b(f) + N_a^b(f) .$$

~~Theorem: A function of~~

Def: $f \in BV[a,b]$ if $T_a^b(f) < \infty$.

Theorem: $f \in BV[a,b] \Leftrightarrow f$ is a difference of two ~~monoton~~ non-decreasing functions ~~on~~ $[a,b] \to \mathbb{R}$

Proof: $\Rightarrow$

$$f(x) = \underbrace{f(a) + P_a^x(f)} - \underbrace{N_a^x(f)}$$

these functions are non-decreasing and finite $(\leq T_a^b(f))$

$\Leftarrow$

Let $f = g - h$, g, h non-decreasing, then

$$t_v(f) = \sum_i |f(x_i) - f(x_{i-1})|$$

$$\leq \sum_i |g(x_i) - g(x_{i-1})| + \sum_i |h(x_i) - h(x_{i-1})|$$

$$= g(b) - g(a) + h(b) - h(a).$$

$$\Rightarrow T_a^b(f) \leq g(b) + h(b) - g(a) - h(a).$$

<u>Corollary</u>: BV-functions are differentiable almost everywhere.

<u>Theorem</u>

Let $\cancel{\forall}$ $f \in L^1([a,b])$, and set

$$F(x) = \int_a^x f(t)\,dt$$

then

(1) F is in $C([a,b]) \cap BV$ $\Rightarrow$ F is diff. a.e.

(2) $F'(x) = f(x)$ for a.e. x in $[a,b]$

Proof Assume $f \geq 0$, otherwise $f = f_+ - f_-$

(1) Continuity follows from absolute continuity of the integral: $\forall_{\varepsilon > 0} \exists_{\delta} \; |A| < \delta \Rightarrow \left|\int_A f\right| < \varepsilon$.

$$|x-y| < \delta \Rightarrow |F(x) - F(y)| = \left|\int_x^y f(t)\,dt\right| < \varepsilon.$$

BV: For any partition of $[a,b]$

$$\sum_{i=1}^{k} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{k} \left| \int_{x_{i-1}}^{x_i} f(t)\,dt \right| \leq$$

$$\leq \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} |f(t)|\,dt = \int_a^b |f(t)|\,dt.$$

$$\Rightarrow T_a^b(F) \leq \int_a^b |f|$$

(2)

We need ~~two facts~~:

Lemma 1: If $f \in L^1([a,b])$ and $\forall_{x \in [a,b]} \int_a^x f(t)\,dt = 0$, then $f(x) = 0$ a.e.

EXERCISE

~~Lemma 2~~

Now suppose first f is bounded on $[a,b]$. by some K

$$|f_n(x)| = \left| \frac{F(x+1/n) - F(x)}{1/n} \right|$$

$$= \left| n \int_x^{x+\frac{1}{n}} f(t)\,dt \right| < M$$

Clearly, $f_n(x) \xrightarrow[\text{pointwise}]{} F'(x)$ a.e

thus, by bounded conv. theorem, for $c < b$

$$\int_a^c F' = \lim_n \int_a^c f_n = \lim_{n\to\infty} n \int_a^c \left(F(x+\tfrac{1}{n}) - F(x)\right) dx$$

$$= \lim_{n\to\infty} n \int_c^{c+\frac{1}{n}} F(t)\,dt - \lim_{n\to\infty} n \int_a^{a+\frac{1}{n}} F(t)\,dt$$

the same calc. as before

$$= F(c) - F(a) \quad \text{since } F \text{ is continuous.}$$

$$= \int_a^c f$$

thus, $\forall_{x \in [a,b)}$ $\int_a^x (F' - f) = 0 \underset{\text{Lemma 1}}{\Rightarrow} F' = f$ a.e

General case of unbounded f:

Let $f_n(x) = \begin{cases} f(x) & f(x) < n \\ n & f(x) \geq n \end{cases}$

then $G_n(x) = \int_a^x (f - f_n)$ is non-decreasing

$\Rightarrow$ differentiable a.e, $\frac{d}{dx} G_n(x) \geq 0$.

$$F'(x) = \frac{d}{dx} \int_a^x f(t)\,dt = \frac{d}{dx} \int_a^x (f - f_n) + \frac{d}{dx} \int_a^x f_n$$

$$= \underbrace{\frac{d}{dx} G_n}_{\geq 0} + \underbrace{\frac{d}{dx} \int_a^x f_n}_{= f_n \text{ a.e}} \geq f_n(x) \text{ a.e}$$

$\Rightarrow F'(x) \geq f(x)$ a.e. (n is arbitrary)

$\Rightarrow \int_a^b F'(x) \geq \int_a^b f(x) = F(b) - F(a)$.

but $f \geq 0 \Rightarrow F$ non-decreasing

$$\Rightarrow \int_a^b F'(x) \leq F(b) - F(a)$$

thus $\int_a^b F'(\cancel{x})\,dt = F(b) - F(a) = \int_a^{b^{*}} f(t)\,dt$

$\Rightarrow F' = f$ a.e.

Def:

f is Absolutely Continuous (AC) on $[a,b]$

if ~~for any~~ $\forall_{\varepsilon>0}\ \exists_{\delta>0}$ s.t.

for any finite family of disjoint intervals $\{I_j\}_{j=1}^{m}$

s.t $\displaystyle\sum_{j=1}^{m} |\underbrace{I_j}_{[\alpha_j,\beta_j]}| < \delta$ we have $\displaystyle\sum \underbrace{\Delta^f_{I_j}}_{|f(\beta_j)-f(\alpha_j)|} < \varepsilon$.

Absolute continuity of the integral

$\Downarrow$

$$F(x) = \int_a^x f(t)\,dt \text{ is in AC if } f \in L^1.$$

$F \in AC \Rightarrow F \in BV$
(Exercise)

$\Rightarrow$ $F \in AC$, then F diff. a.e.

Task: $F \in AC\,[a,b]$ iff $\exists_{f \in L^1}$

$$F(x) = \cancel{f}\ F(a) + \int_a^x f(t)\,dt$$

Lemma: $f \in AC[a,b]$ and $f'(x) = 0$ a.e

$\Rightarrow f$ is constant.

Proof: Choose $c \in [a,b]$

Let $E \subset [a,b]$ be a set of full measure such that $f'(x) = 0$ $\forall_{x \in E}$.

$$f'(x) = 0 \iff \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0$$

$$\Downarrow$$

$\forall_{\eta > 0}$ there exist arbitrarily short intervals $[x, x+h]$ such that $|f(x+h) - f(x)| < \eta h$

Such intervals cover E; ~~for a Vitali co~~ they form a Vitali covering

$\Rightarrow$ we can choose a finite family of disjoint intervals I_i in $[a,b]$ $\quad I_i = [x_i, y_i]$ $\quad i = 1, \dots, n$

$$y_0 = a \le x_1 < y_1 < x_2 < y_2 < \dots < x_n < y_n \le b = x_{n+1}$$

such that $\cancel{\sum_{i=1}^{m} (y_i - x_i)}$ $\left| [a,c] \setminus \bigcup I_i \right| = \sum_{k=0}^{n} |x_{i+1} - y_i| < \delta$

where δ corresponds to ε in the defn. of A.C.

$$\Rightarrow \sum_{i=1}^{n} |f(x_{i+1}) - f(y_i)| < \varepsilon$$

$$\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \eta \sum_{i=1}^{n} (y_i - x_i)$$

$\Rightarrow f(c) - f(a) \le \varepsilon + \eta (c - a)$; ε, η arb. small $\Rightarrow f(c) = f(a)$

<u>Proof of theorem</u>

What is left to prove is that if $F \in AC[a,b]$, then $\exists\, f \in L^1[a,b]$

$$F(x) = F(a) + \int_a^x f(t)\,dt .$$

$F \in AC \Rightarrow F \in BV \Rightarrow \exists\, F_1, F_2$ non decreasing

$F = F_1 - F_2$; F' exists a.e,

$$|F'(x)| \leq F_1'(x) + F_2'(x) \qquad (\Delta\text{-ineq} + F_1', F_2' \geq 0)$$

$$\Rightarrow \int_a^x |F'(t)|\,dx \leq \int_a^x \left(F_1'(t) + F_2'(t)\right) \leq F_1(x) + F_2(x) - F_1(a) - F_2(a)$$
$$\leq (F_1+F_2)(b) - (F_1+F_2)(a)$$

$\Rightarrow F'$ is integrable on $[a,b]$

$$G(x) = \int_a^x F'(t) \quad \text{is } AC,$$

$$g(x) = F(x) - G(x) \quad \text{is also } AC,$$

$$g'(x) = 0 \quad \text{a.e.} \Rightarrow g(x) = \text{const}$$

$$g(x) = g(a) = F(a) - 0 = F(a).$$