We obtained Sobolev's inequality

$$\|u - u_\Omega\|_{L^q(\Omega)} \leq C(n,p,\Omega)\, \|\nabla u\|_{L^p(\Omega)}$$

with $q = \dfrac{np}{n-p}$

but, as I mentioned, the proof was not valid for $p=1$ (we used Hardy-Littlewood-Sobolev fractional integration theorem).

We need another approach.

<u>Theorem</u> (Sobolev inequality on cubes, for $p=1$)

For any $f \in W^{1,1}(Q)$, $Q \subset \mathbb{R}^n$ a cube,

we have $\|f\|_{L^{n/n-1}(Q)} \overset{(*)}{\leq} \|\nabla f\|_{L^1(Q)} + |Q|^{-1/n} \|f\|_{L^1(Q)}$

(for $n=1$ LHS reads $\|f\|_{L^\infty(Q)} \leq \ldots$).

<u>Proof</u>: As usual, we prove the inequality for smooth f and proceed to $f \in W^{1,1}$ by density.

Moreover, it is enough to prove $(*)$ for $Q = [0,1]^n$ the unit cube

<u>Exercise</u>: Check that if $f : Q \to \mathbb{R}^n$, A is the affine map ($Ax = \lambda x + x_0$) that takes $[0,1]^n$ to Q ($A([0,1]^n) = Q$), $g(x) = f(Ax)$, then $g : [0,1]^n \to \mathbb{R}$

and if $\|g\|_{L^{n/n-1}([0,1]^n)} \leq \|\nabla g\|_{L^1([0,1]^n)} + \|g\|_{L^1([0,1]^n)}$,

then f satisfies $(*)$.

From now on, $f \in C^\infty(Q)$, $Q = [0,1]^n$.

Proof goes by induction on the dimension n.

$\boxed{n=1}$ f is smooth, thus there exists $y \in (0,1)$ s.t. $|f(y)| = \fint_{[0,1]} |f(s)|\,ds = \int_0^1 |f(s)|\,ds = \|f\|_{L^1}$

Now, for any $t \in [0,1]$

$$f(t) = \int_y^t f'(s)\,ds + f(y)$$

thus

$$|f(t)| \leq \left|\int_y^t f'(s)\,ds\right| + |f(y)|$$

$$\leq \left|\int_y^t |f'(s)|\,ds\right| + \|f\|_{L^1}$$

$$\leq \int_0^1 |f'(s)|\,ds + \|f\|_{L^1} = \|f'\|_{L^1} + \|f\|_{L^1}$$

$$\Longrightarrow \|f\|_\infty \leq \|f'\|_{L^1} + \|f\|_{L^1}$$

$\boxed{\text{inductive step}}$

Assume that for $\tilde{Q} = [0,1]^{n-1}$ and any smooth ~~$f: Q$~~ $h: \tilde{Q} \to \mathbb{R}$

$$\|h\|_{L^{\frac{n-1}{n-2}}(\tilde{Q})} \leq \|\nabla f\|_{L^1(\tilde{Q})} + \|f\|_{L^1(\tilde{Q})}$$

understood as ∞ for $n=2$

$$Q = \underset{t}{[0,1]} \times \underset{y}{\tilde{Q}}$$

by result for $n=1$

$$|f(t,y)| \leq \int_0^1 \left|\frac{\partial f}{\partial t}(\tau,y)\right| d\tau + \int_0^1 |f(\tau,y)|\, d\tau.$$

Integrate y over $\tilde{Q}$:

$$\|f(t,\cdot)\|_{L^1(\tilde{Q})} = \int_{\tilde{Q}} |f(t,y)|\, dy \leq \int_{\tilde{Q}} \int_0^1 \left|\frac{\partial f}{\partial t}(\tau,y)\right| d\tau\, dy +$$

$$+ \int_{\tilde{Q}} \int_0^1 |f(\tau,y)|\, d\tau\, dy$$

$$\overset{(*)}{\leq} \|\nabla f\|_{L^1(Q)} + \|f\|_{L^1(Q)}.$$

On the other hand

$$\int_{\tilde{Q}} |f(t,y)|^{\frac{n}{n-1}} dy = \int_{\tilde{Q}} |f|^{\frac{1}{n-1}} |f| \overset{\text{Hölder}}{\leq} \left(\int_{\tilde{Q}} |f|\right)^{\frac{1}{n-1}} \left(\int_{\tilde{Q}} |f|^{\frac{n-1}{n-2}}\right)^{\frac{n-2}{n-1}}$$

$$= \|f(t,\cdot)\|_{L^1(\tilde{Q})}^{1/n-1} \|f(t,\cdot)\|_{L^{\frac{n-1}{n-2}}(\tilde{Q})}$$

$$\overset{\text{inductive assumption}}{\leq} \|f(t,\cdot)\|_{L^1(\tilde{Q})}^{\frac{1}{n-1}} \left(\|f(t,\cdot)\|_{L^1(\tilde{Q})} + \left\|\frac{\partial f}{\partial y}(t,\cdot)\right\|_{L^1(\tilde{Q})}\right)$$

~~We Estimate~~ $$\underset{\text{by } (*)}{\leq} \left(\|f\|_{L^1(Q)} + \|\nabla f\|_{L^1(Q)}\right)^{\frac{1}{n-1}} \left(\|f(t,\cdot)\|_{L^1(\tilde{Q})} + \left\|\frac{\partial f}{\partial y}(t,\cdot)\right\|_{L^1(\tilde{Q})}\right)$$

Integrate the inequality with t from 0 to 1:

$$\|f\|_{L^{n/n-1}(Q)}^{n/n-1} \leq \left(\|f\|_{L^1(Q)} + \|\nabla f\|_{L^1(Q)}\right)^{\frac{1}{n-1}} \left(\|f\|_{L^1(Q)} + \|\nabla f\|_{L^1(Q)}\right)$$

$$= \left(\|f\|_{L^1(Q)} + \|\nabla f\|_{L^1(Q)}\right)^{\frac{n}{n-1}}$$

which immediately yields the desired inequality.

Definition A domain $\Omega \subset \mathbb{R}^n$ has no external cusps if there exists $A \in (0,1]$ such that for any $R \in (0, \operatorname{diam} \Omega]$ and any $x \in \overline{\Omega}$

$$|\Omega \cap B(x,R)| \overset{(*)}{\geq} A\,|B(x,R)| = A R^n \omega_n .$$

Exercise : Prove that a domain with cone property has no external cusps.

We shall write $\tilde{B}(x,R)$ for $\Omega \cap B(x,R)$;

note that $|\tilde{B}(x,R)| \underset{\substack{\uparrow \\ \text{up to a constant } (A)}}{\approx} |B(x,R)| = R^n \omega_n$.

Exercise Prove that if Ω has no ext. cusps, then for any $x \in \overline{\Omega}$, $s, r \in (0, \operatorname{diam} \Omega]$

$$A\left(\frac{s}{r}\right)^n |\tilde{B}(x,r)| \leq |\tilde{B}(x,s)| \leq A^{-1}\left(\frac{s}{r}\right)|\tilde{B}(x,r)|$$

Theorem (Lebesgue's differentiation theorem)
For any $f \in L^1_{loc}(\mathbb{R}^n)$ we have

$$\lim_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)|\,dx = 0 \quad \text{for a.e. } x.$$

Exercise : Deduce from the above theorem that if $g(x) = \lim_{r \to 0} \fint_{B(x,r)} f(t)\,dt$, then $g(x) = f(x)$ for a.e. x.

Some more notation : $u_{x,r} = \fint_{\tilde{B}(x,r)} u = u_{\tilde{B}(x,r)}$

Theorem (Campanato)
Let Ω be a domain with no external cusps.
Suppose $u \in L^p(\Omega)$ satisfies, for some $\lambda \in (n, n+p)$,

$$[u]_{p,\lambda}^p := \sup \left\{ R^{-\lambda} \int_{\tilde{B}(x,R)} |u - u_{x,R}|^p \ : \ x \in \overline{\Omega},\ R \in (0, \operatorname{diam} \Omega] \right\} < \infty .$$

Then there exists a Hölder continuous function $v : \overline{\Omega} \to \mathbb{R}$, with Hölder exponent $\alpha = \frac{\lambda - n}{p}$, such that $u = v$ a.e. in Ω.

Exercise: Deduce Morrey's theorem
Suppose Ω has the cone property, $p > n$ and $f \in W^{1,p}(\Omega)$. Then there exists a Hölder continuous function g, with Hölder exponent $\gamma = 1 - \frac{n}{p}$, such that $f = g$ a.e.

Proof of Campanato's theorem
We shall need two lemmata. The first one tells us how to estimate the difference of mean values of u, centered at the same point:

Lemma 1 Let $\tau = \frac{\alpha}{n} = \frac{\lambda - n}{np}$. There exists $C = C(\Omega, n, \lambda, p)$ such that for any $x \in \overline{\Omega}$ and all r, s: $0 \le r \le s \le \operatorname{diam} \Omega$ we have

$$|u_{x,s} - u_{x,r}| \le C [u]_{p,\lambda}\, s^{\alpha}$$

Proof of Lemma 1

To shorten the notation, I shall write u_s and u_r for $u_{x,s}$ and $u_{x,r}$ and so on, since x is fixed.

We have, for any $0 < \rho \leq \sigma \leq \operatorname{diam} \Omega$,

$$|u_\rho - u_\sigma| = \left| \fint_{\tilde{B}(x,\rho)} (u - u_\sigma) \right| \leq \fint_{\tilde{B}(x,\rho)} |u - u_\sigma|$$

$$\overset{\text{Hölder}}{\leq} \left(\fint_{\tilde{B}(x,\rho)} |u - u_\sigma|^p \right)^{1/p} = |\tilde{B}(x,\rho)|^{-\frac{1}{p}} \left(\int_{\tilde{B}(x,\rho)} |u - u_\sigma|^p \right)^{1/p}$$

$$\leq C \rho^{-\frac{n}{p}} \left(\sigma^\lambda [u]^p_{p,\lambda} \right)^{1/p} = C \rho^{-\frac{n}{p}} \sigma^{\frac{\lambda}{p}} [u]_{p,\lambda}$$

Now, we choose a sequence of radii:

$s_1 = s$, $s_2 = \frac{1}{2} s$, ..., $s_k = s/2^{k-1}$

such that $s_k \geq r > s_{k+1}$. Then $\frac{r}{s} \in [2^{-k}, 2^{1-k})$.

We estimate

$$|u_s - u_r| \leq |u_s - u_{s_{k+1}}| + |u_r - u_{s_{k+1}}|$$

$$\leq \sum_{j=1}^{k} |u_{s_j} - u_{s_{j+1}}| + |u_r - u_{s_{k+1}}|$$

$$\leq \sum_{j=1}^{k} C s_j^{-\frac{n}{p}} s_{j+1}^{\frac{\lambda}{p}} [u]_{p,\lambda} + C r^{-\frac{n}{p}} s_{k+1}^{\frac{\lambda}{p}} [u]_{p,\lambda}$$

$$= C [u]_{p,\lambda} s^{\frac{\lambda - n}{p}} \left(\sum_{j=1}^{k} 2^{(j-1)\frac{n}{p} - j\frac{\lambda}{p}} + \left(\frac{r}{s}\right)^{-\frac{n}{p}} 2^{-k\frac{\lambda}{p}} \right)$$

$$\leq C [u]_{p,\lambda} s^{\alpha} \left(2^{-n/p} \sum_{j=1}^{k} 2^{-j\alpha} + 2^{-k\alpha} \right)$$

$$\leq C [u]_{p,\lambda} s^{\alpha} \underbrace{\left(1 + 2^{-n/p} \sum_{j=1}^{\infty} 2^{-j\alpha} \right)}_{\text{constans.}}$$

Lemma 2 estimates the difference of mean values centered at two different points.

Lemma 2 Suppose $x, y \in \Omega$, $|x-y| = R$

Then $|u_{x,2R} - u_{y,2R}| \le C R^{\cancel{\frac{\lambda-n}{p}}\alpha} [u]_{p,\lambda}$

for some $C = C(\Omega, n, p, \lambda)$

Proof: Take any $z \in \tilde{B}(x,2R) \cap \tilde{B}(y,2R)$.

We have

$$|u_{x,2R} - u_{y,2R}| \le |u_{x,2R} - u(z)| + |u(z) - u_{y,2R}|$$

Averaging the above over $\tilde{B}(x,2R) \cap \tilde{B}(y,2R)$ we get

$$|u_{x,2R} - u_{y,2R}| \le \fint_{\tilde{B}(x,2R)\cap\tilde{B}(y,2R)} |u(z) - u_{x,2R}| + \fint_{\tilde{B}(x,2R)\cap\tilde{B}(y,2R)} |u(z) - u_{y,2R}| = I_1 + I_2$$

Note that $\tilde{B}(x,R) \subset \tilde{B}(x,2R) \cap \tilde{B}(y,2R) \subset \tilde{B}(x,2R)$, thus

$$I_1 = \fint_{\tilde{B}(x,2R)\cap\tilde{B}(y,2R)} |u(z) - u_{x,2R}| \le \frac{1}{|\tilde{B}(x,R)|} \int_{B(x,2R)} |u(z) - u_{x,2R}|$$

$$\le C(A,n) R^{-n} \int_{B(x,2R)} |u(z) - u_{x,2R}|$$

$$\cancel{\le C(A,n) \fint_{B(x,2R)} |u(z) - u_{x,2R}|}$$

$$\le C(A,n) \fint_{B(x,2R)} |u(z) - u_{x,2R}| \overset{\text{Hölder}}{\le} C(A,n) \left(\fint_{B(x,2R)} |u(z) - u_{x,2R}|^p \right)^{1/p}$$

$$= C(A,n,\lambda,p) R^{\frac{\lambda-n}{p}} \left((2R)^{-\lambda} \int_{B(x,2R)} |u(z) - u_{x,2R}|^p \right)^{1/p}$$

$$\le C(A,n,\lambda,p) R^{\alpha} [u]_{p,\lambda}.$$

In the same way we estimate I_2, which yields the lemma.

Proof of Campanato's theorem

Set $v(x) = \lim_{r \to 0} \fint_{\tilde{B}(x,r)} u(y)\,dy = \lim_{r\to 0} u_{x,r}$

It follows from Lebesgue's differentiation theorem that $v(x) = u(x)$ for a.e $x \in \Omega$.

We want to estimate $|v(x) - v(y)|$ for some $x, y \in \overline{\Omega}$, $|x-y| = R$.

$$
\begin{aligned}
|v(x) - v(y)| &\le |v(x) - v_{x,2R}| + |v_{x,2R} - v_{y,2R}| \\
&\quad + |v_{y,2R} - v(y)| \\
&= B + D + E
\end{aligned}
$$

By Lemma 1,

$$
\begin{aligned}
B = |v(x) - v_{x,2R}| &= \lim_{r \to 0} \fint |u_{x,r} - u_{x,2R}| \\
&\le C\,[u]_{p,\lambda}\,(2R)^\alpha
\end{aligned}
$$

and E likewise.

$$
\begin{aligned}
D = |v_{x,2R} - v_{y,2R}| &= |u_{x,2R} - u_{y,2R}| \\
&\le C\,[u]_{p,\lambda} \cdot R^\alpha \quad \text{by Lemma 2.}
\end{aligned}
$$

Altogether,
$$
\begin{aligned}
|v(x) - v(y)| &\le C\,[u]_{p,\lambda}\,R^\alpha \\
&= C\,[u]_{p,\lambda}\,|x-y|^\alpha
\end{aligned}
$$

which proves our assertion.

Gagliardo - Nirenberg inequality

1. Let $u \in C_0^\infty(\mathbb{R}^n)$, then

$$\|\nabla u\|^2_{L^2} = \int_{\mathbb{R}^n} \langle \nabla u, \nabla u \rangle = - \int \Delta u \cdot u$$

thus

$$\|\nabla u\|^2_{L^2} \overset{\text{Hölder}}{\le} \|\Delta u\|_{L^2} \cdot \|u\|_{L^2} \le \|D^2 u\|_{L^2} \cdot \|u\|_{L^2}$$

or $\le \|\Delta u\|_{L^p} \cdot \|u\|_{L^q} \le \|D^2 u\|_{L^p} \cdot \|u\|_{L^q}$

where $\frac{1}{p} + \frac{1}{q} = 1$

in particular for $p = 1$

$$\|\nabla u\|^2_{L^2} \le \|D^2 u\|_{L^1} \cdot \|u\|_{L^\infty}.$$

This of course extends to all the Sobolev spaces where these norms are finite

2. More generally

<u>Theorem</u>: For $1 \le p \le k$, $u \in C_0^\infty(\mathbb{R}^n)$,

$$\|D_j u\|^2_{L^{2k/p}} \le C \|u\|_{L^{2k/p-1}} \|D_j^2 u\|_{L^{2k/p+1}}$$

<u>Proof</u>: Let $q = \frac{2k}{p}$. For any $v \in C_0^\infty(\mathbb{R}^n)$

$$D_j (v |v|^{q-2}) = (q-1) |v|^{q-2} D_j v$$

We shall apply this to $v = D_j u$:

$$|D_j u|^q = \sum D_j u \cdot D_j u \cdot |D_j u|^{q-2} =$$

$$= D_j (u D_j u |D_j u|^{q-2}) - u D_j (D_j u |D_j u|^{q-2})$$

$$= D_j(u D_j u |D_j u|^{q-2}) - (q-1) u |D_j u|^{q-2} D_j^2 u$$

Integrating over all $\mathbb{R}^n$ we get

$$\int_{\mathbb{R}^n} |D_j u|^q = \underbrace{\int_{\mathbb{R}^n} D_j(u D_j u |D_j u|^{q-2})}_{=0, \text{ since } u \in C_0^\infty(\mathbb{R}^n)} - (q-1)\int u |D_j u|^{q-2} D_j^2 u$$

$$\leq (q-1) \|u\|_{\frac{2k}{p-1}} \cdot \|D_j u\|_q^{q-2} \cdot \|D_j^2 u\|_{\frac{2k}{p+1}}$$

$\uparrow$ Hölder with exponents $\frac{2k}{p-1}, \frac{q}{q-2}, \frac{2k}{p+1}$

$$\frac{p-1}{2k} + \frac{p+1}{2k} + \frac{q-2}{q} =$$

$$= \frac{p}{k} + 1 - \frac{2}{q} = 1, \text{ since } q = \frac{2k}{p}.$$

Dividing both sides by $\|D_j u\|_q^{q-2}$ we obtain the desired inequality.

Hölder inequality works all right also in the limit case $p = 1$, where we get

$$\|D_j u\|_{L^{2k}}^2 \leq C \|u\|_{L^\infty} \cdot \|D_j^2 u\|_{L^k}$$

In particular, suppose $u \in W^{2,2}(\mathbb{R}^n)$, $n > 4$, then, by Sobolev's imbedding, $\nabla u \in W^{1,\frac{2n}{n-2}}(\mathbb{R}^n)$ and $\frac{2n}{n-2} < n$, so there is no chance for $u \in L^\infty$ in general, nor for $\nabla u \in L^4(\mathbb{R}^n)$ $\left(2 \cdot \frac{n}{n-2} < 4\right)$.

However, if we KNOW that $u \in L^\infty$, then, by G-N,

$$\|\nabla u\|_{L^4}^2 \leq \|u\|_{L^\infty} \cdot \|D^2 u\|_{L^2}$$

$$\Rightarrow \quad L^\infty(\mathbb{R}^n) \cap W^{2,2}(\mathbb{R}^n) \subset W^{1,4}(\mathbb{R}^n).$$

Likewise, for $n > 2$, $L^\infty(\mathbb{R}^n) \cap W^{2,1}(\mathbb{R}^n) \subset W^{1,2}(\mathbb{R}^n)$,

even though $D^2 u \in L^1 \Rightarrow \nabla u \in L^{\frac{n}{n-1}}$

$$\Downarrow$$

$$u \in L^{\frac{n}{n-2}}$$

Trudinger's inequality

Suppose $B = B(0,1) \subset \mathbb{R}^n$, $u \in W^{1,n}(B)$, $n > 1$

By Sobolev's embedding theorem, $u \in L^p(B)$ for any $1 \leq p < \infty$.

What can we say about the sequence $\left(\|u\|_{L^p(B)} \right)_p$?

If the sequence were bounded:

$\exists_M \ \forall_{p>1} \ \|u\|_{L^p} \leq M$,

we would have, by Chebychev's inequality

$$|\{x \in B : |u(x)| > \lambda\}| \leq \frac{\|u\|_{L^p}^p}{\lambda^p} \leq \frac{M^p}{\lambda^p} \xrightarrow[\text{if } \lambda > M]{} 0$$

thus $u \in L^\infty(B)$, $\|u\|_{L^\infty} \leq M$.

This cannot be true in general – we know that $W^{1,n}(B)$ contains unbounded functions.

<u>Theorem</u> (Trudinger, Moser). Let u be as above. Then.

$$\|u - u_B\|_{L^p} \leq C(n)\, p^{\frac{n-1}{n}} \|\nabla u\|_{L^n}.$$

Proof: To simplify notation, we shall write $u - u_B = f$. We want to estimate $\|f\|_p$; let $g \in \underset{\substack{\| \\ (L^p(B))^*}}{L^q(B)}$, $\frac{1}{p} + \frac{1}{q} = 1$.

Thanks to the representation formula, we have

$$|f(x)| = |u(x) - u_B| \leq C(n) \int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}} dy = C(n) \int_B \frac{|\nabla f(y)|\, dy}{|x-y|^{n-1}},$$

thus

$$\int_B |fg| \leq C(n) \int_B \int_B \frac{|\nabla f(y)|\,|g(x)|}{|x-y|^{n-1}} dy\, dx$$

$$= C(n) \int_B \int_B \frac{|\nabla f(y)|\,|g(x)|^{\frac{1}{n}}}{|x-y|^{\frac{n-1}{np}}} \cdot \frac{|g(x)|^{\frac{n-1}{n}}}{|x-y|^{\frac{(np-1)(n-1)}{np}}} dx\, dy$$

Hölder with exponents n & $\frac{n}{n-1}$

$$\leq C(n) \underbrace{\left[\int_B \int_B \frac{|\nabla f(y)|^n |g(x)|}{|x-y|^{\frac{n-1}{p}}} dx\, dy \right]^{1/n}}_{I_1^{1/n}} \underbrace{\left[\int_B \int_B \frac{|g(x)|\, dx\, dy}{|x-y|^{n-\frac{1}{p}}} \right]^{\frac{n-1}{n}}}_{I_2^{\frac{n-1}{n}}} = (*)$$

Exercise: Let $y \in \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, $|\Omega| < \infty$. Then, for any $\alpha \in (0, n)$,

$$\int_\Omega \frac{dx}{|x-y|^{n-\alpha}} \leq n \omega_n^{1-\frac{\alpha}{n}} \frac{|\Omega|^{\frac{\alpha}{n}}}{\alpha}$$

(for $\Omega = B(0,1)$ $\quad \int_B \frac{dx}{|x-y|^{n-\alpha}} \leq \frac{n\omega_n}{\alpha}$)

With the help of the exercise, we shall estimate I_1 and I_2 separately

$$I_2 = \int_B\int_B \frac{|g(x)|}{|x-y|^{n-\frac{1}{p}}}\,dy\,dx = \int_B |g(x)| \left[\int_B \frac{dy}{|x-y|^{n-\frac{1}{p}}}\right] dx$$

$$\leq \int_B |g(x)| \cdot n\omega_n^{1-\frac{1}{np}} \frac{\cancel{|\cdot|}^{\frac{1}{np}}\ \omega_n^{\frac{1}{np}}}{1/np} \leq n^2 \omega_n^{1+\frac{1}{p}} \cdot p\, \|g\|_q$$

$$I_1 = \int_B\int_B \frac{|\nabla f(y)|^n |g(x)|}{|x-y|^{\frac{n-1}{p}}}\,dx\,dy = \int_B |\nabla f(y)|^n \cdot \left[\int_B \frac{|g(x)|}{|x-y|^{\frac{n-1}{p}}}\,dx\right] dy$$

$$\underset{\text{with exp. } q,p}{\overset{\text{Hölder in the inner integral}}{\leq}} \int_B |\nabla f(y)|^n \|g\|_q \left(\int_B \frac{dx}{|x-y|^{n-1}}\right)^{\frac{1}{p}} dy$$

$$\underset{\text{Exercise}}{\leq} \|\nabla f\|_n^n \cdot \|g\|_q \cdot n^{\frac{1}{p}} \omega_n^{\frac{1}{p}}$$

Combining the estimates, we have

$$\int_B |fg| \leq (*) = \cancel{k}_{\geq 1}\, C(n)\, I_1^{1/n} I_2^{\frac{n-1}{n}} \leq \cancel{C(n)(n\omega_n)^{\frac{n}{p}\frac{1}{np}} \|g\|_q^{1/n} \|\nabla f\|}$$

$$\leq C(n) \cdot (n\omega_n)^{\frac{1}{np}} \cdot \|\nabla f\|_n^{\cancel{n}} \|g\|_q^{\frac{1}{n}} \cdot n^{\frac{2(n-1)}{n}} \omega_n^{\frac{n-1}{n}(1+\frac{1}{p})} p^{\frac{n-1}{n}} \|g\|_q^{\frac{n-1}{n}}$$

$$\leq \tilde{C}(n)\, p^{\frac{n-1}{n}} \|\nabla f\|_n \|g\|_q$$

And this holds for ANY $g \in L^q(B)$. Therefore

$$\|f\|_{L^p} = \|f\|_{(L^q)^*} \leq \tilde{C}(n)\, p^{\frac{n-1}{n}} \|\nabla f\|_n .$$

$\square$

<u>Corollary</u>: (the real Trudinger's theorem)
There exists $\tilde{C} = \tilde{C}(n)$ such that, for any $u \in W^{1,n}(B)$,

$$\int_B \exp\left[\left(\frac{|u-u_B|}{\tilde{C}\,\|\nabla u\|_n}\right)^{\frac{n}{n-1}}\right] \leq 1$$

<u>Remark</u>: The set of functions $f \in L^1(B)$ such that there exists $0 < A \in \mathbb{R}$ with

$$(*) \int_B \exp\left(\left|\frac{f}{A}\right|^{n/n-1}\right) \leq 1,$$

equipped with norm $\|\cdot\|_{L^1} + \cancel{\#} A_0$, where A_0 is the infimum of A satisfying $(*)$), forms the so-called Orlicz space with Orlicz function $\exp(t^{n/n-1}) - 1$. This is a Banach space that is larger than L^∞, but contains $\bigcap_{q>1} L^q$. ~~for~~ ~~...~~.

<u>Proof of ~~#~~ Trudinger's theorem</u>

Let us write out Trudinger's inequality with $p = k \cdot \frac{n}{n-1}$:

$$\|u-u_B\|_{k\cdot\frac{n}{n-1}} \leq C(n)\left(k\cdot\frac{n}{n-1}\right)^{\frac{n-1}{n}} \|\nabla \cancel{f}\, u\|_n, \text{ equivalently}$$

$$\int_B \left(|u-u_B|^{\frac{n}{n-1}}\right)^k \leq \left(k\cdot\frac{n}{n-1}\right)^k \cdot \left[C(n)\,\|\nabla \cancel{f}\, u\|_n\right]^{k\cdot\frac{n}{n-1}}$$

We want to estimate

$$(*) = \int_B \exp\left(\left|\frac{u-u_B}{A}\right|^{\frac{n}{n-1}}\right)$$

and find A such that $(*) \leq 1$, $A = \cancel{\tilde{C}(n)\|\nabla u\|}$
$= \tilde{C}(n) \cdot \|\nabla u\|_n$

$$(*) = \sum_{k=0}^{\infty} \frac{1}{k!} \int_B \left|\frac{u-u_B}{A}\right|^{k \cdot \frac{n}{n-1}}$$

$$\leq \sum_{k=0}^{\infty} \frac{1}{k!} \left[k \cdot \frac{n}{n-1} \left(C(n) \frac{\|\nabla u\|_n}{A}\right)^{\frac{n}{n-1}}\right]^k = (**)$$

Time for some trivial exercises in Calculus:

1. $\sum_{k=0}^{\infty} \frac{k^k}{3^k k!}$ is convergent (d'Alemberts ratio test)

   Denote its sum by S.

2. $\sum_{k=0}^{\infty} \left(\frac{k}{3S}\right)^k \cdot \frac{1}{k!} \leq 1$

3. We shall now take A in $(**)$ such that

$$\frac{n}{n-1} \left(C(n) \frac{\|\nabla u\|_n}{A}\right)^{\frac{n}{n-1}} = \frac{1}{3S}$$

$$\Rightarrow A = \frac{C(n)}{\left(\frac{n-1}{n}\right)^{n-1/n} (3S)^{n-1/n}} \|\nabla u\|_n = \tilde{C}(n) \|\nabla u\|_n$$

Then, of course, $(**) \leq 1$.

BMO space

bounded mean oscillation

$u \in L^1_{loc}(\mathbb{R}^n)$ is in $BMO(\mathbb{R}^n)$ if there exists $A \geq 0$ such that for any ball $B \subset \mathbb{R}^n$

$$\fint_B |u - u_B| \leq A.$$

The least such A is the BMO-(semi)-norm of u, denoted $\|u\|_{BMO}$.

Of course, $\|u\|_{BMO} = 0 \iff u = \text{const}$.

Remarks:

One can define equivalently the BMO space using cubes, or cubes with edges parallel to the axes, in place of balls. We can also localize the definition, to get $BMO(\Omega)$: $u \in BMO(\Omega)$ if $u \in L^1_{loc}(\Omega)$ & $\fint_B |u - u_B| < A$ for any ball $B \subset \Omega$.

Observations

$L^\infty(\mathbb{R}^n) \subsetneq BMO(\mathbb{R}^n)$ (and the same for local versions).

$u \in L^\infty(\mathbb{R}^n)$

$\Downarrow$

$$\fint_B |u - u_B| \leq 2\|u\|_\infty$$

Exercises

1. Prove that $\log|x| \in BMO(\mathbb{R}^n)$

2. Assign to any ball $B \subset \mathbb{R}^n$ a constant c_B and define

$$\widetilde{BMO}(\mathbb{R}^n) = \Big\{ f \in L^1_{loc}(\mathbb{R}^n) : \cancel{\fint_{B} |u - c_B| <} \ \exists_A \forall_B \fint_B |u - c_B| < A \Big\}$$

(i.e. the same definition ~~of~~ as for BMO, but with c_B in place of u_B).

Prove that $\widetilde{BMO}(\mathbb{R}^n) \subseteq BMO(\mathbb{R}^n)$.

Theorem: $W^{1,n}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$.

Likewise, ~~if Ω has the cone property~~

$$W^{1,n}(\Omega) \subset BMO(\Omega).$$

Proof:

All we need is Poincaré's inequality on balls:

$$\fint_B |u - u_B| \leq C(n) \cdot r \fint_B |\nabla u| \leq C(n)\, r \left(\fint_B |\nabla u|^n\right)^{1/n}$$

$B = B_r$

$$\leq C(n) \left(\int_B |\nabla u|^n\right)^{1/n}$$

$$\leq C(n)\, \|\nabla u\|_{L^n(\mathbb{R}^n)\ (\Omega)}.$$

□.

In fact, we proved more:

$$\fint_B |u-u_B| \le C(n)\, \|\nabla u\|_{L^n(B)} \xrightarrow{\text{radius of } B \to 0} 0$$

The subspace of ~~functions~~ BMO containing the functions, for which the oscillation $\fint_B |u-u_B|$ on balls B is not merely bounded, but tends to 0 with radius of B, is called the space VMO (vanishing mean oscillation).

All continuous functions belong to VMO. (obvious).

BMO has numerous interesting properties, I shall give some of them without proofs, as they would take easily a half-semester course.

Facts on BMO functions

1. John, Nirenberg:

There exist $c_1(n)$, $c_2(n)$ such that for any ~~ball~~ $u \in BMO(\Omega)$ and for any ball $B \subset \Omega$

$$|\{x \in B : |u(x) - u_B| > t\}| \leq c_1 \exp\left(\frac{-c_2 t}{\|u\|_{BMO}}\right)|B|$$

Corollary: $u \in BMO(\Omega) \Rightarrow u \in L^p_{loc}(\Omega)$

2. The pre-dual space to $BMO(\mathbb{R}^n)$ is well understood:

The real Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ has numerous equivalent (Fefferman, Stein) definitions

1. ~~$f \in L^1(\mathbb{R}^n)$ is in $\mathcal{H}^1(\mathbb{R}^n)$~~

~~Fix~~ Fix ~~$\varphi \in C_0^\infty(B$~~ $\varphi \in C_0^\infty(B(0,1))$, $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi = 1$

and form a standard mollifyier:
$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$.

~~$\varphi \in L^1(\mathbb{R}^n$~~

$$\mathcal{H}^1(\mathbb{R}^n) = \left\{ f \in L^1(\mathbb{R}^n) : \sup_{\varepsilon > 0} |f * \varphi_\varepsilon| \in L^1 \right\}$$

2. A function $a \in L^\infty(\mathbb{R}^n)$ is an <u>atom</u>

if (1) $\exists$ ball $B \subset \mathbb{R}^n$ $\operatorname{supp} a \subset B$

(2) $|a(x)| \le \frac{1}{|B|}$ a.e.

(3) $\int_B a = \int_{\mathbb{R}^n} a = 0$.

$f \in \mathcal{H}^1(\mathbb{R}^n) \iff f \overset{a.e.}{=} \sum_{k=1}^{\infty} \lambda_k a_k$, with a_k - atoms $\cancel{\sum_{k=1}^{\infty} |\lambda_k| < \infty}$

and $f \in L^1(\mathbb{R}^n)$.

3. Define Riesz transforms

$R_j \quad (j = 1, 2, \dots, n)$

$R_j f(x) = \text{p.v.}\ f * \frac{x_i}{|x|^{n+1}}$

(or, equivalently, $(R_j f)^\wedge(\xi) = i \frac{\xi_i}{|\xi|} \hat{f}$)

Then $f \in \mathcal{H}^1(\mathbb{R}^n) \iff f \in L^1(\mathbb{R}^n)$ & $R_j f \in L^1$ for $j = 1, \dots, n$.

$(VMO(\mathbb{R}^n))^* = \mathcal{H}^1(\mathbb{R}^n)$

$(\mathcal{H}^1(\mathbb{R}^n))^* = BMO(\mathbb{R}^n)$ Fefferman

Coifman, Lions, Meyer, Semmes $\quad \frac{1}{p}+\frac{1}{q}=1$

Suppose $u \in W^{1,p}(\mathbb{R}^n)$, $E \in L^q(\mathbb{R}^n,\mathbb{R}^n)$, $\varphi \in BMO(\mathbb{R}^n)$

and $\int_{\mathbb{R}^n} E\cdot\nabla \xi = 0$ for any $\xi \in C_0^\infty(\mathbb{R}^n)$

$(\operatorname{div} E = 0)$.

Then

$$\int_{\mathbb{R}^n} \langle E, \nabla u\rangle \xi \le C(n,p) \|E\|_{L^q}\|\nabla u\|_{L^p}\|\xi\|_{BMO}$$

This is stronger than (known earlier)

Wente's inequality: let E, u as above and suppose $u \in W^{1,n}$, then

$$\int_{\mathbb{R}^n}\langle E,\nabla u\rangle \xi \le C(n,p)\|E\|_{L^q}\|\nabla u\|_{L^p}\|\nabla\xi\|_{L^n}$$

~~Meyer, Riviere~~ Optimal Gagliardo–Nirenberg inequalities

Meyer, Riviere: $\quad B$ or Ω, smooth bdd domain

$$\|\nabla f\|^2_{L^4(B)} \le C\cdot\|f\|_{BMO(B)}\|f\|_{W^{2,2}(B)}$$

Strzelecki $\quad f\in W^{k,p}(\mathbb{R}^n)$, $p>1$, $1\le m<k$

$$\|D^m f\|_{L^{\frac{kp}{m}}} \le C(n)\|f\|_{BMO}^{1-\frac{m}{k}}\|D^k f\|_{L^p}^{\frac{m}{k}}$$