Lebesgue spaces

Take $p>0$, $E\subset\mathbb{R}^n$, $|E|>0$.

We say that a measurable $f:E\to\mathbb{R}$ is in the Lebesgue space $L^p(E)$ iff $\int_E |f|^p<\infty$.

- this is a linear space:

for $p\geq 1$

$$\int_E |\alpha f+\beta g|^p \leq 2^{p-1}\Big[|\alpha|^p\int_E|f|^p+|\beta|^p\int_E|g|^p\Big]$$

$$\Big(\underbrace{|x+y|^p\leq 2^{p-1}(|x|^p+|y|^p)}_{\text{Jensen's inequality}}\Big)$$

for $p\in(0,1)$

even easier: $|x+y|^p\leq|x|^p+|y|^p$ (why?)

thus

$$\int_E|\alpha f+\beta g|^p\leq|\alpha|^p\int_E|f|^p+|\beta|^p\int_E|g|^p$$

- is this a normed space?

A natural candidate for the norm:

$$\|f\|_p=\Big(\int_E|f|^p\Big)^{1/p}.$$

for $p \geq 1$ $\|\cdot\|_p$ satisfies the triangle inequality (which is Minkowski's inequality) and, of course, $\|\lambda f\|_p = |\lambda| \, \|f\|_p$.

However, if $0 \not\equiv f = 0$ $\cancel{p}$ a.e., then

$\|f\|_p = 0$ — there exist non-zero (barely non-zero, though) functions s.t. their p-norm is 0.

Therefore, ~~on $M(E)$~~ $\|\cdot\|_p$ is only a semi-norm.

We remedy this situation by introducing an equivalence relation

$$f \sim g \quad \text{iff} \quad f = g \text{ a.e. on } E.$$

Then $f \sim g \iff \|f-g\|_p = 0$.

The elements of the Lebesgue space L^p are not (measurable) functions, but equivalence classes of $\sim$. We shall neglect this and use the former terminology, tacitly identifying functions that agree a.e.

For $p \in (0,1)$ the problem is more grave, since $\|\cdot\|_p$ does not satisfy the triangle inequality. In fact, we have

$$\| |f| + |g| \|_p \geq \|f\|_p + \|g\|_p \quad \left(\begin{array}{c}\text{reverse Minkowski} \\ \text{inequality}\end{array}\right)$$

(exercise).

However, $L^p(E)$, understood as a space of equivalence classes, is a metric space, with $d(f,g) = \|f-g\|_p^p$.

We shall rarely use L^p spaces for $p<1$, and concentrate on the case $1 \leq p < \infty$, when $L^p(E)$ is a Banach space.

↑

to be proven soon.

We shall yet introduce the $L^\infty(E)$ space of essentially bounded functions

$$L^\infty(E) = \{\underbrace{[f] \in M(E)/\sim}_{\text{equiv. classes of } \sim} : \operatorname*{esssup}_E |f| < \infty\}$$

where

$$\|f\|_\infty = \operatorname*{esssup}_E |f| = \inf\{\alpha : |\{x \in E : |f(x)| > \alpha\}| = 0\}$$

$$= \inf\{\sup_{E \setminus A} |f| : |A| = 0\}.$$

Theorem: (Frigyes Riesz, Fischer)
The space $L^p(E)$ with metric $d(f,g) = \|f-g\|_p$ is, for any $p \in [1,\infty]$, complete.

Remark. Also for $p \in (0,1)$, the space $L^p(E)$, equipped with metric $d(f,g) = \|f-g\|_p^p$, is complete; the proof follows the same lines as the proof of R.-F. Theorem.

Lemma: A normed linear space V is complete if and only if every ~~unconditionally conv~~ absolutely convergent series of elements of V is convergent in V

$$\left(\text{i.e. } \sum_{k=1}^{\infty} \|a_k\| < \infty \Rightarrow \exists \lim_{m\to\infty} \sum_{k=1}^{m} a_k\right)$$

Proof of the lemma
$\Rightarrow$ (in a complete space all abs. convergent series are convergent).

Suppose $\sum_{k=1}^{\infty} f_k$ is an absolutely ~~continuous~~ convergent series. Then, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $\sum_{k=N}^{\infty} \|f_k\| < \varepsilon$.

Set $S_m = \sum_{k=1}^{m} f_k$ and suppose $m > l > N$.

Then $\| S_m - S_l \| = \| \sum_{k=l+1}^{m} f_k \| \leq \sum_{k=l+1}^{m} \| f_k \|$

$$\leq \sum_{k=l+1}^{\infty} \| f_k \| < \varepsilon,$$

which shows that (S_m) is a Cauchy sequence, therefore it is convergent.

$\Leftarrow$ (the other way)

Suppose (a_m) is a Cauchy sequence in V:

$$\forall_k \exists_{m_k} \forall_{s,l > m_k} \quad | a_s - a_l | < 2^{-k}.$$

We can assume that $m_{k+1} > m_k$ for all k.

The sequence $(a_{m_k})_{k \in \mathbb{N}}$ is a subsequence of (a_m); let $b_1 = a_{m_1}$, $b_2 = a_{m_2} - a_{m_1}$, ...,

$$b_k = a_{m_k} - a_{m_{k-1}}.$$

Of course $\sum_{l=1}^{k} b_l = a_{m_k}$, moreover $\forall_{k>1} \; \| b_k \| < 2^{-k+1}$, thus the series $\sum_{l=1}^{\infty} b_l$ is absolutely convergent. Together with our assumption that an abs. convergent series is convergent we get (a_{m_k}) convergent, as a sequence, to some $a \in V$.

Convergence of the whole sequence (a_m) is immediate:

$$\|a_m - a\| \le \underbrace{\|a_m - a_{m_k}\|}_{\substack{\text{small for large} \\ m \text{ and } k \\ \text{by Cauchy's condition}}} + \underbrace{\|a_{m_k} - a\|}_{\substack{\text{small for large } k \\ \text{by convergence} \\ \text{of } a_{m_k} \text{ to } a.}}$$

<u>Proof of Riesz-Fischer Theorem</u>

Suppose $\sum_{m=1}^{\infty} f_m$ is an absolutely convergent series in $L^p(E)$.

We set $g_m(x) = \sum_{k=1}^{m} |f_k(x)|$.

The function $g_m(x)$ is measurable, but it might take infinite (i.e. $+\infty$) values.

For every $m \in \mathbb{N}$ we have

$$\|g_m\|_p \underset{\substack{\text{triangle} \\ \text{inequality}}}{\le} \sum_{k=1}^{m} \|f_k\|_p \le \sum_{k=1}^{\infty} \|f_k\|_p =: M,$$

thus $\int_E |g_m|^p \le M^p$.

Also, for every $x \in E$, the sequence $(g_m(x))$ is a non-increasing sequence of numbers from ~~$\mathbb{R} \cup \{+\infty\}$~~ $[0,\infty]$, thus it is convergent to some $g(x) \in [0,\infty]$.

The function $g: E \to [0,\infty]$ is measurable, and by monotone convergence theorem

$\int_E g^p \leq M^p$, thus g is in L^p, in particular

$g(x)$ is ~~finite~~ for a.e. $x \in E$.

Equivalently, the series $\sum_{k=1}^{\infty} |f_k(x)|$ (of real numbers) is absolutely convergent for a.e. $x \in E$

$\Rightarrow$ it is convergent for a.e. $x \in E$

Setting $$f(x) = \begin{cases} \sum_{k=1}^{\infty} f_k(x) & \text{if } g(x) < \infty \\ 0 & \text{if } g(x) = +\infty, \end{cases}$$

we get a measurable function that is a.e. a pointwise ~~limit of~~ sum of the series $\sum_{k=1}^{\infty} f_k$.

Note also that $\forall_{m \in \mathbb{N}} \quad |\sum_{k=1}^{m} f_k(x)| \leq g_m(x) \leq g(x)$

thus also in the limit $|S(x)| \leq g(x)$ (everywhere!)

We therefore have

$$\forall_{x \in E} \quad |\sum_{k=1}^{m} f_k(x) - S(x)|^p \leq 2^p [g(x)]^p$$

By Dominated Convergence Theorem

$$\lim_{m\to\infty} \int_E \left|\sum_{k=1}^{m} f_k(x) - S(x)\right|^p = 0$$

which proves that $\sum_{k=1}^{m} f_k \xrightarrow{L^p} S$

$\Rightarrow$ the series $\sum_{k=1}^{\infty} f_k$ is convergent in L^p.

$\square$.

Remark: The proof is written for $p<\infty$, but it is in fact even easier for $p=\infty$.

A closer inspection of the proof yields the following

Theorem: Out of every sequence (f_m) convergent in $L^p(E)$, $1\leq p\leq\infty$, we can choose a subsequence that is a.e. pointwise convergent.

Proof: For $p = \infty$ this is rather trivial; (why?)

The sequence (f_m), being convergent, satisfies Cauchy's condition, thus, like in the proof of the lemma, we can choose a subsequence (f_{m_k}) s.t.

$$\| f_{m_k} - f_{m_{k+1}} \| \leq 2^{-k}$$

We have

$$f_{m_l}(x) = f_{m_1}(x) + \sum_{k=1}^{l} \left(f_{m_{k+1}}(x) - f_{m_k}(x) \right).$$

The series $\sum (f_{m_{k+1}} - f_{m_k})$ is absolutely convergent in L^p, and, exactly as in the proof of R.-F. Theorem, it is convergent for a.e. $x \in E$. This yields a.e. convergence of $\cancel{f_m}$ $f_{m_l}(x)$.

Dual space to L^p

Theorem For any $p \in (1,\infty)$ the dual space (i.e. the space of all bounded linear functionals) of $L^p(E)$ is $L^q(E)$, where $q = \frac{p}{p-1}$ $\left(\frac{1}{q} + \frac{1}{p} = 1\right)$.

Remarks

1. The theorem holds also for $p=1$ (dual to L^1 is L^∞), as long as the underlying measure is tame enough (e.g. σ-finite). For $p \in (1,\infty)$, the result is valid for any measure. We, however, deal with Lebesgue measure only, so all's well.

2. The theorem fails for $p = \infty$
$(L^\infty)^*$ consists of all finitely-additive signed measures that are absolutely continuous w.r. to Lebesgue measure.

However, constucting an example of a functional on L^∞ that is not given by
$f \longmapsto \int fg$ for some $g \in L^1$

requires non-constructive techniques like the construction of Banach limits via Hahn-Banach theorem.

Before we start talking about the proof, let us first look closer at functionals on L^p given by integration against an L^q-function.

For $p \in [1,\infty]$, we set $q = \begin{cases} \frac{p}{p-1} & p \in (1,\infty) \\ 1 & p = \infty \\ \infty & p = 1 \end{cases}$

and, for a given $g \in L^q$

$$T_g(f) = \int_E fg$$

By Hölder's inequality,

$$|T_g(f)| = \left|\int_E fg\right| \leq \|f\|_p \|g_E\|_q ,$$

thus it is indeed a bounded functional on L^p,

$$|||T_g|||_{(L^p)^*} = \|T_g\| := \sup_{\|f\|_p \leq 1} |T_g(f)| \leq \|g\|_q$$

Taking $f = |g|^{q-1}\operatorname{sgn} g$ we see that

$$|f|^p = |g|^{p(q-1)} = |g|^q \in L^1, \text{ thus } f \in L^p$$

$$\|f\|_p = \|g\|_q^{q/p}$$

$$T_g(f) = \int_E fg = \int |g|^q = \|g\|_q^q = \|g\|_q^{q/p} \cdot \|g\|_q$$

$$= \|f\|_p \, \|g\|_q$$

which shows that in fact

$$\|T_g\| = \|g\|_q .$$

This works for $p = \infty$, $q = 1$ as well, just take $f = \operatorname{sgn} g \in L^\infty$

for any fixed representative of $g \in L^1$.

<u>$p = 1$, $q = \infty$</u> Exercise: show that, again, $\|T_g\| = \|g\|_\infty$.

Our exercises show that for any $p \in [1,\infty]$ 13
$L^q \subset (L^p)^*$, and what remains to prove is that for $p \in [1,\infty)$ every bounded functional on L^p has the form T_g for some $g \in L^q$.

<u>Outline of a standard proof</u> (see e.g. Rudin's Real & Complex Analysis)

Let $T \in (L^p)^*$

Suppose first that <u>$|E| < \infty$</u>.

- for any measurable $A \subset E$ set $\lambda(A) = T(\chi_A)$
- show that λ is countably additive (it is finitely additive by linearity of T)

$\Rightarrow \lambda$ is a (signed) measure on E

- show that λ is absolutely continuous with respect to Lebesgue measure

$\Rightarrow$ by Radon-Nikodym theorem $\exists\, g \in L^1$ s.t. $\lambda(A) = T(\chi_A) = \int_A g = \int_E \chi_A g$

- by linearity of T

$$T(f) \overset{(*)}{=} \int_E f g$$

holds for all simple functions f

- any L^∞ function is a uniform limit of simple functions; uniform convergence implies L^p-convergence

$\Rightarrow$ $(*)$ holds for all L^∞-functions

~~L^q is dense in L^p~~

- evaluating T on $f_m = sgn(g) \cdot |g|^{q-1} \cdot \chi_{\{|g| \le m\}}$ yields $g \in L^q$

- $T(\cdot)$ agrees with $T_g(\cdot)$ on a dense subset $L^\infty \subset L^p$, by continuity

$$T(f) = T_g(f) = \int_E f g$$

for all $f \in L^p$.

If $|E| = \infty$, then $E = \bigsqcup_i E_i$, $|E_i| < \infty$

countable disjoint sum

we prove the theorem in every E_i and glue the resulting g_i's.

A non-measure-theoretic proof (for $p \in (1,\infty)$)

Def. A normed linear space X is uniformly convex iff

$$\forall_{\varepsilon>0} \ \exists_{\delta>0} \ \forall_{x,y\in X} \quad \|x\| \le 1,\ \|y\| \le 1,\ \|x-y\| \ge \varepsilon \Downarrow \left\|\frac{x+y}{2}\right\| \le 1-\delta.$$

Lemma: Let (x_n) be a sequence in X, X uniformly convex. If $\|x_n\| \le 1$ and $\|x_n + x_m\| \xrightarrow[n,m\to\infty]{} 2$, then (x_n) is a Cauchy sequence.

Proof: Fix $\varepsilon > 0$ and take $\delta > 0$ s.t. uniform convexity condition holds.

$$\|x_n + x_m\| \xrightarrow[n,m\to\infty]{} 2 \Longleftrightarrow \exists_{n_0} \ \forall_{n,m>n_0} \ \left\|\frac{x_n+x_m}{2}\right\| > 1-\delta,$$

therefore, by uniform convexity, $\|x_n - x_m\| < \varepsilon$ for $n,m > n_0$.

<u>Def</u>: A sequence (x_n) of elements of a normed linear space X is <u>weakly convergent</u> to $x \in X$ iff for every bounded linear functional $f : X \to \mathbb{R}$ we have $f(x_n) \to f(x)$. We denote it by $x_n \rightharpoonup x$.

<u>Theorem</u>: Let X be a uniformly convex Banach space. If (x_n), a sequence of elements of X, is weakly convergent to x and $\|x_n\| \xrightarrow[n\to\infty]{} \|x\|$, then $x_n \to x$ (i.e. $\|x_n - x\| \to 0$)

<u>Proof</u>: Nothing to do if $x=0$.

If $x \neq 0$, then for n suff. large $x_n \neq 0$.

Set $y_n = \frac{x_n}{\|x_n\|}$, $y = \frac{x}{\|x\|}$.

For any $g \in X^*$ $\quad g(y_n) = g\left(\frac{x_n}{\|x_n\|}\right) = \frac{g(x_n)}{\|x_n\|} \to \frac{g(x)}{\|x\|} = g(y)$

thus $y_n \rightharpoonup y$.

Let $f \in X^*$ s.t. $\|f\|_{X^*} = 1$, $f(y) = 1$ (Hahn-Banach)

$$f\left(\frac{y_n+y_m}{2}\right) \leq \left\|\frac{y_n+y_m}{2}\right\| \leq 1$$

$$\| $$

$$\frac{1}{2}\left(f(y_n)+f(y_m)\right)$$

$$\Big\downarrow n, m \to \infty$$

$$f(y) = 1$$

$$\Rrightarrow$$

$$\left\|\frac{y_n+y_m}{2}\right\| \longrightarrow 1$$

$\Downarrow$ by Lemma

(y_n) is a Cauchy sequence.

~~We~~ X is a Banach space, thus

y_n is convergent (to some $\tilde{y}$) in X.

Strong (norm) convergence implies weak convergence and weak limits are unique, thus $\tilde{y} = y$

$$\|y_n - y\| \to 0$$

$$\|x_n - x\| \leq \|x_n\| \underbrace{\|y_n - y\|}_{\to 0} + \|y\| \underbrace{(\|x_n\| - \|x\|)}_{\to 0} \longrightarrow 0.$$

$\square$.

<u>Fact</u>: Every uniformly convex Banach space X is reflexive, i.e. $(X^*)^* = X$

<u>Theorem</u> : L^p spaces are, for $p\in(1,\infty)$, uniformly convex.

<u>Lemma</u> (Clarkson's inequalities)

(1) for $p\in[2,\infty)$ and $f,g\in L^p$

$$\|f+g\|_p^p+\|f-g\|_p^p\le 2^{p-1}\left(\|f\|_p^p+\|g\|_p^p\right)$$

(2) for $p\in(1,2]$, $q=\frac{p}{p-1}$ and any $f,g\in L^p$

$$\|f+g\|_p^q+\|f-g\|_p^q\le 2\left(\|f\|_p^p+\|g\|_p^p\right)^{q-1}$$

<u>Proof</u> – Exercise course

<u>Proof of Theorem</u>

<u>$p\in[2,\infty)$</u> : $f,g\in L^p$, $\|f\|_p\le 1$, $\|g\|_p\le 1$, $\|f-g\|_p\ge\varepsilon$, then

$$\left\|\frac{f+g}{2}\right\|_p^p=2^{-p}\|f+g\|_p^p\overset{(1)}{\le}2^{-p}\,2^{p-1}\left(\|f\|_p^p+\|g\|_p^p\right)-\|f-g\|_p^p$$
$$\le 1-\varepsilon^p$$

thus $\left\|\frac{f+g}{2}\right\|_p\le(1-\varepsilon^p)^{1/p}$; we take $\delta=1-(1-\varepsilon^p)^{1/p}>0$.

$p \in (1,2]$

$$\left\| \frac{f+g}{2} \right\|_p^q = 2^{-q} \| f+g \|_p^q \leq 2^{-q}\left(2\left(\|f\|_p^p + \|g\|_p^p \right)^{q-1} - \right.$$
$$\left. - \|f-g\|_p^q \right)$$
$$\leq 2^{-q}\left(2 \cdot 2^{q-1} - \varepsilon^q\right) = 1 - \left(\frac{\varepsilon}{2}\right)^q$$

and we can take $\delta = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^q\right)^{1/q}$.

<u>Lemma (McShane)</u>

Let T be a bounded linear functional on a normed linear space X.
Suppose that for some $f, g \in X$

(•) $\|g\| = 1$ and $T(g) = \|T\|_{X^*}$

and (••) $\lim\limits_{t \to 0} \dfrac{\|g+tf\|^p - \|g\|^p}{pt}$ exists for some $p \geq 1$.

Then $T(f) = \|T\| \cdot \lim\limits_{t \to 0} \dfrac{\|g+tf\|^p - \|g\|^p}{pt}$

Proof of McShane's Lemma

First, note that

$$\lim_{t\to 0} \frac{(T(g+tf))^p - (T(g))^p}{pt} =$$

$$= \lim_{t\to 0} \frac{(T(g)+tT(f))^p - (T(g))^p}{pt}$$

$$= T(g)^{p-1} T(f) = \|T\|^{p-1} T(f).$$

$$\left\{ \begin{aligned} &\lim_{t\to 0} \frac{(a+bt)^p - a^p}{pt} = \\ &= \frac{1}{p} \frac{d}{dt}\Big|_{t=0} (a+bt)^p \\ &= \frac{1}{p} \cdot p a^{p-1} \cdot b \\ &= a^{p-1} \cdot b \end{aligned} \right.$$

Next, $\|T\| \, \|g\| = T(g)$

$\|T\| \, \|g+tf\| \geq T(g+tf)$

thus

$$\lim_{t\to 0^+} \|T\|^p \frac{\|g+tf\|^p - \|g\|^p}{pt} \geq$$

$$\geq \lim_{t\to 0^+} \frac{(T(g+tf))^p - (T(g))^p}{pt}$$

$$= \|T\|^{p-1} T(f)$$

$$= \lim_{t\to 0^-} \frac{(T(g+tf))^p - (T(g))^p}{pt}$$

$$\geq \|T\|^p \frac{\|g+tf\|^p - \|g\|^p}{pt}$$

By $(\cdot\cdot)$, both ends are equal, thus

$$T(f) = \|T\| \lim_{t\to 0} \frac{\|g+tf\|^p - \|g\|^p}{pt}$$

Proof of $L^p - L^q$ duality

Scheme : ① for any $T \in (L^p)^*$ find $g \in L^p$ ($\neq 0$ ← $0 = T$ not interesting)

s.t. $\|g\|_p = 1$, $T(g) = \|T\|$

② compute the limit from (••), McShane's Lemma

① Choose a sequence (g_n) in L^p s.t.

$\|g_n\| = 1$, $T(g_n) \longrightarrow \|T\| \neq 0$

Then $2 \geq \|g_n + g_m\|_p \geq \dfrac{|T(g_n + g_m)|}{\|T\|} \xrightarrow[n,m]{} 2$

thus, by Lemma, (g_n) is a Cauchy sequence

$$g_n \xrightarrow{L^p} g \Rightarrow \underset{\swarrow \atop \|T\|}{T(g_n)} \longrightarrow T(g) \Rightarrow T(g) = \|T\|.$$

②. $\dfrac{d}{dt}\Big|_{t=0} |a+bt|^p = p|a+bt|^{p-1} \cdot b \cdot \operatorname{sgn}(a+bt)\Big|_{t=0} =$

$= p|a|^{p-1} b \operatorname{sgn} a$

$$\lim_{t\to 0} \frac{\|g+tf\|_p^p - \|g\|_p^p}{pt} \underset{\uparrow}{=} \int_E \lim_{t\to 0} \frac{|g+tf|^p - |g|^p}{pt} =$$

Dominated Convergence Theorem

$$= \int_E \frac{\cancel{p}|g|^{p-1} f \operatorname{sgn} g}{\cancel{p}} = \int_E |g|^{p-1} \operatorname{sgn} g \cdot f$$

Therefore, for every $f \in L^p$

$$T(f) = \|T\| \int_E f |g|^{p-1} \operatorname{sgn} g = T_h(f)$$

for $h = \|T\| \cdot |g|^{p-1} \operatorname{sgn} g$.