Lebesgue spaces

Take $p > 0$, $E \subset \mathbb{R}^n$, $|E| > 0$. We say that a measurable $f : E \to \mathbb{R}$ is in the Lebesgue space $L^p(E)$ if $\int_E |f|^p < \infty$.

- this is a linear space:

  $\int_E |\alpha f + \beta g|^p \leq 2^{p-1} \left( |\alpha|^p \int_E |f|^p + |\beta|^p \int_E |g|^p \right)$

  for $p \geq 1$

  $(1 + \frac{1}{p}) \leq 2^{p-1} (1 + \frac{1}{p})$  (Jensen's inequality)

  for $p \in (0, 1)$

even easier: $|x+y|^p \leq |x|^p + |y|^p$ (why?)

thus

$\int_E |\alpha f + \beta g|^p \leq |\alpha|^p \int_E |f|^p + |\beta|^p \int_E |g|^p$

- is this a normed space?

A natural candidate for the norm:

$\|f\|_p = \left( \int_E |f|^p \right)^{1/p}$.
for $p > 1$, $\| \cdot \|_p$ satisfies the triangle inequality (which is Minkowski's inequality) and, of course, $\| \lambda f \|_p = |\lambda| \|f\|_p$.

However, if $0 \neq f = 0$ a.e., then

$$\|f\|_p = 0$$

- there exist non-zero (barely non-zero, though)
  functions s.t. their $p$-norm is 0.

Therefore, on $L^p$, $\| \cdot \|_p$ is only a semi-norm.

We remedy this situation by introducing an equivalence relation

$$f \sim g \iff f = g \text{ a.e. on } E.$$  

Then

$$f \sim g \iff \|f - g\|_p = 0.$$  

The elements of the Lebesgue space $L^p$ are not (measurable) functions, but equivalence classes of $\sim$. We shall neglect this and use the former terminology, tacitly identifying functions that agree a.e.
For $p \in (0, 1)$ the problem is more grave, since $\|f\|_p$ does not satisfy the triangle inequality. In fact, we have

$$\|f\|_p + \|g\|_p \geq \|f + g\|_p \quad (\text{reverse Minkowski inequality})$$

(exercise).

However, $L^p(E)$, understood as a space of equivalence classes, is a metric space, with $d(f,g) = \|f - g\|_p$.

We shall rarely use $L^p$ spaces for $p < 1$, and concentrate on the case $1 \leq p \leq \infty$, when $L^p(E)$ is a Banach space.

\[ \uparrow \]

\[ \text{to be proven soon.} \]

We shall yet introduce the $L^\infty(E)$ space of essentially bounded functions

$$L^\infty(E) = \left\{ \left[ f \right] \in M(E)^\sim : \text{esssup}_{E} |f| < \infty \right\}$$

where

$$\|f\|_\infty = \text{esssup}_{E} |f| = \inf_{\{A \subseteq E : |f(x)| > \alpha \} = \emptyset} \{\alpha \}$$

$$= \inf_{\{A \subseteq E : |A| = 0\}} \sup_{E \setminus A} |f|$$
Theorem (Frigyes Riesz, Fischer)

The space $L^p(E)$ with metric $d(f,g) = \|f - g\|_p$ is, for any $p \in [1, \infty]$, complete.

Remark. Also for $p \in (0,1)$, the space $L^p(E)$, equipped with metric $d(f,g) = \|f - g\|_p^p$, is complete; the proof follows the same lines as the proof of R.-F. Theorem.

Lemma: A normed linear space $V$ is complete if and only if every absolutely convergent series of elements of $V$ is convergent in $V$.

(i.e. $\sum_{k=1}^{\infty} \|a_k\| < \infty \Rightarrow \exists \lim_{m \to \infty} \sum_{k=1}^{m} a_k$)

Proof of the lemma

$\Rightarrow$ (in a complete space all abs. convergent series are convergent).

Suppose $\sum_{k=1}^{\infty} f_k$ is an absolutely continuous convergent series. Then, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $\sum_{k=N}^{\infty} \|f_k\| < \varepsilon$. 

\[ \sum_{k=1}^{m} a_k \]
Set $S_m = \sum_{k=1}^{m} f_k$ and suppose $m > l > N$.

Then $\| S_m - S_l \| = \| \sum_{k=l+1}^{m} f_k \| \leq \sum_{k=l+1}^{\infty} \| f_k \| < \varepsilon,$

which shows that $(S_m)$ is a Cauchy sequence, therefore it is convergent.

$\Leftarrow$ (the other way)

Suppose $(a_m)$ is a Cauchy sequence in $V$: For all $\varepsilon > 0$, there exists $m_k \in \mathbb{N}$ such that $\forall m, n > m_k$, $|a_m - a_n| < 2^{-k}$.

We can assume that $m_{k+1} > m_k$ for all $k$.

The sequence $(a_{m_k})_{k\in\mathbb{N}}$ is a subsequence of $(a_m)$; let $b_1 = a_{m_1}$, $b_2 = a_{m_2} - a_{m_1}$, ..., $b_k = a_{m_k} - a_{m_{k-1}}$.

Of course $\sum_{l=1}^{k} b_l = a_{m_k}$, moreover $\forall k > 1$, $\| b_k \| < 2^{-k+1}$, thus the series $\sum_{k=1}^{\infty} b_k$ is absolutely convergent.

Together with our assumption that an abs. convergent series is convergent we get $(a_{m_k})$ convergent, as a sequence, to some $a \in V.$
Convergence of the whole sequence \((a_m)\) is immediate:

\[
\|a_m - a\| \leq \|a_m - a_{m_k}\| + \|a_{m_k} - a\|
\]
small for large \(m\) and \(k\) by convergence
small for large \(k\) by Cauchy’s condition of \(a_{m_k}\) to \(a\).

\textbf{Proof of Riesz–Fischer Theorem}

Suppose \(\sum_{m=1}^{\infty} f_m\) is an absolutely convergent series in \(L^p(E)\).
We set \(g_m(x) = \sum_{k=1}^{m} |f_k(x)|\).

The function \(g_m(x)\) is measurable, but it might take infinite (i.e., \(+\infty\)) values.

For every \(m \in \mathbb{N}\) we have

\[
\|g_m\|_p \leq \sum_{k=1}^{m} \|f_k\|_p \leq \sum_{k=1}^{\infty} \|f_k\|_p =: M,
\]

triangle inequality

thus \(\int_E |g_m|^p \leq M^p\).

Also, for every \(x \in E\), the sequence \((g_m(x))\) is a non-increasing sequence of numbers from \(\mathbb{R} \cup \{\infty\}\), thus it is convergent to some \(g(x) \in [0, \infty]\).
The function \( g: E \rightarrow [0, \infty] \) is measurable, and by monotone convergence theorem
\[
\sum_{k=1}^{\infty} f_k(x) \leq g(x)
\]
for a.e. \( x \in E \).

Equivalently, the series \( \sum_{k=1}^{\infty} |f_k(x)| \) (of real numbers) is absolutely convergent for a.e. \( x \in E \).

\[\Rightarrow \text{it is convergent for a.e. } x \in E \]

Setting
\[
f(x) = \begin{cases} \sum_{k=1}^{\infty} f_k(x) & \text{if } g(x) < \infty \\ 0 & \text{if } g(x) = +\infty \end{cases}
\]

we get a measurable function that is a.e. a pointwise limit of sum of the series \( \sum_{k=1}^{\infty} f_k \).

Note also that
\[
\forall m \in \mathbb{N} \quad \left| \sum_{k=1}^{m} f_k(x) \right| \leq g_m(x) \leq g(x)
\]
thus also in the limit \( |S(x)| \leq g(x) \) (everywhere!).

We therefore have
\[
\forall x \in E \quad \left| \sum_{k=1}^{m} f_k(x) - S(x) \right| \leq 2^p \left[ g(x) \right]^p
\]
By Dominated Convergence Theorem

\[
\lim_{m \to \infty} \int_E \left| \sum_{k=1}^m f_k(x) - S(x) \right|^p = 0
\]

which proves that \( \sum_{k=1}^m f_k \in L^p \) \( \xrightarrow{L^p} S \)

\( \Rightarrow \) the series \( \sum_{k=1}^\infty f_k \) is convergent in \( L^p \).

\[\square\]

Remark: The proof is written for \( p < \infty \), but it is in fact even easier for \( p = \infty \).

A closer inspection of the proof yields the following

Theorem: Out of every sequence \((f_m)\) convergent in \( L^p(E), 1 \leq p \leq \infty \), we can choose a subsequence that is a.e. pointwise convergent.
Proof: For $p = \infty$ this is rather trivial. (why?)

The sequence $(f_m)$, being convergent, satisfies Cauchy's condition, thus, like in the proof of the lemma, we can choose a subsequence $(f_{m_k})$ s.t.

$$||f_{m_k} - f_{m_{k+1}}|| \leq 2^{-k}$$

We have

$$f_{m_k}(x) = f_{m_1}(x) + \sum_{k=1}^{l} (f_{m_{k+1}}(x) - f_{m_k}(x)).$$

The series $\Sigma (f_{m_{k+1}} - f_{m_k})$ is absolutely convergent in $L^p$, and, exactly as in the proof of R.-F. Theorem, it is convergent for a.e. $x \in E$. This yields a.e. convergence of $f_{m_k}$ to $f_m(x)$. 
Dual space to $L^p$

**Theorem** For any $p \in (1, \infty)$ the dual space (i.e. the space of all bounded linear functionals) of $L^p(E)$ is $L^q(E)$, where $q = \frac{p}{p-1}$ ($\frac{1}{q} + \frac{1}{p} = 1$).

**Remarks**

1. The theorem holds also for $p=1$ (dual to $L^1$ is $L^{\infty}$), as long as the underlying measure is tame enough (e.g. $\sigma$-finite). For $p \in (1, \infty)$, the result is valid for any measure. We, however, deal with Lebesgue measure only, so all's well.

2. The theorem fails for $p=\infty$

$(L^\infty)^*$ consists of all finitely-additive signed measures that are absolutely continuous w.r. to Lebesgue measure.

However, constructing an example of a functional on $L^\infty$ that is not given by $f \mapsto f g$ for some $g \in L^1$. 
requires non-constructive techniques like the construction of Banach limits via Hahn-Banach theorem.

Before we start talking about the proof, let us first look closer at functionals on \( L^p \) given by integration against an \( L^q \)-function.

For \( p \in [1, \infty] \), we set \( q = \begin{cases} \left\lfloor \frac{p}{p-1} \right\rfloor & p \in (1, \infty) \\ 1 & p = \infty \\ 1 & p = 1 \end{cases} \)

and, for a given \( g \in L^q \)

\[
T_g(f) = \int f g \quad \text{in } E
\]

By Hölder's inequality,

\[
|T_g(f)| = \left| \int f g \right| \leq \|f\|_p \|g\|_{\frac{p}{q}}
\]

thus it is indeed a bounded functional on \( L^p \),

\[
\|T_g\|_{(L^p)^*} = \|T_g\| = \sup_{\|f\|_p \leq 1} |T_g(f)| \leq \|g\|_q
\]
Taking \( f = |g|^{q-1} \text{sgn} g \), we see that

\[ |f|^p = |g|^p(q-1) = |g|^q \in L^1, \text{ thus } f \in L^p \]

\[ \|f\|_p = \|g\|_{q^p} \]

\[ T_g(f) = \int_E f g = \int_E |g|^q = \|g\|_q^q = \|g\|_q^{q/p} \cdot \|g\|_q \]

which shows that in fact

\[ \|T_g\| = \|g\|_q. \]

This works for \( p = \infty, q = 1 \) as well, just take \( f = \text{sgn} g \in L^\infty \uparrow \)

\[ \text{for any fixed representative of } g \in L^1. \]

\[ p = 1, q = \infty \]

Exercise: show that, again, \( \|T_g\| = \|g\|_\infty. \)
Our exercises show that for any \( p \in [1, \infty] \),
\( L^q \subset (L^p)^* \), and what remains to prove is that for \( p \in [1, \infty) \) every bounded functional on \( L^p \) has the form \( T_g \) for some \( g \in L^q \).

Outline of a standard proof (see e.g. Rudin's Real & Complex Analysis)

Let \( T \in (L^p)^* \)

Suppose first that \( |E| < \infty \).

- For any measurable \( A \subset E \)
  set \( \lambda(A) = T(\chi_A) \)

- Show that \( \lambda \) is countably additive (it is finitely additive by linearity of \( T \))

\( \Rightarrow \) \( \lambda \) is a (signed) measure on \( E \)

- Show that \( \lambda \) is absolutely continuous with respect to Lebesgue measure

\( \Rightarrow \) by Radon-Nikodym theorem \( \exists g \in L^1 \)

s.t. \( \lambda(A) = T(\chi_A) = \int \chi_A g \)

\( \int_E \)
by linearity of \( T \)
\[
T(f) = \int_E f \, g
\]
holds for all simple functions \( f \)

any \( L^\infty \) function is a uniform limit of simple functions; uniform convergence implies \( L^p \)-convergence

\( \Rightarrow \) \((\ast)\) holds for all \( L^\infty \)-functions

\( L^2 \) is dense in \( L^p \)

evaluating \( T \) on \( f_m = \text{sgn}(g) \cdot |g|^{q-1} \cdot g \)
yields \( g \in L^q \)

\( T(f) \) agrees with \( T_g(\cdot) \) on a dense subset \( L^\infty \subset L^p \), by continuity

\[
T(f) = T_g(f) = \int_E f \, g
\]
for all \( f \in L^p \).

If \( |E| = \infty \), then \( E = \bigcup E_i \), \( |E_i| < \infty \)

we prove the theorem in every \( E_i \) and

we glue the resulting \( g_i \)'s.
A non-measure-theoretic proof
(for $p \in (1, \infty)$)

Def. A normed linear space $X$ is uniformly convex iff

$$\forall \exists \delta > 0 \quad \forall x, y \in X \quad \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \quad \Rightarrow \quad \|\frac{x + y}{2}\| \leq 1 - \delta.$$  

Lemma: Let $(x_n)$ be a sequence in $X$, $X$ uniformly convex. If $\|x_n\| \leq 1$ and $\|x_n + x_m\| \rightarrow 2$, then $(x_n)$ is a Cauchy sequence.

Proof: Fix $\varepsilon > 0$ and take $\delta > 0$ s.t. uniform convexity condition holds.

$$\|x_n + x_m\| \rightarrow 2 \iff \exists \forall n_0, m, > n_0 \quad \|\frac{x_n + x_m}{2}\| > 1 - \delta,$$

therefore, by uniform convexity, $\|x_n - x_m\| < \varepsilon$ for $n, m > n_0$. 
Theorem: Let $X$ be a uniformly convex Banach space. If $(x_n)$, a sequence of elements of $X$, is weakly convergent to $x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$ (i.e. $\|x_n - x\| \to 0$).

Proof: Nothing to do if $x = 0$.
If $x \neq 0$, then for $n$ suff. large $x_n \neq 0$.
Set $y_n = \frac{x_n}{\|x_n\|}$, $y = \frac{x}{\|x\|}$.

For any $g \in X^*$, $g(y_n) = g\left(\frac{x_n}{\|x_n\|}\right) = \frac{g(x_n)}{\|x_n\|} \to \frac{g(x)}{\|x\|}$
thus $y_n \to y$.

Let $f \in X^*$ s.t. $\|f\|_{X^*} = 1$, $f(y) = 1$.
\[
\frac{f(y_n + y_m)}{2} \leq \frac{y_n + y_m}{2} \leq 1
\]
\[
\frac{1}{2} \left( f(y_n) + f(y_m) \right) \downarrow_{n,m \to \infty} f(y) = 1
\]
\[
\frac{\|y_n + y_m\|}{2} \to 1
\]

\[\Rightarrow \text{ by Lemma} \]

\((y_n)\) is a Cauchy sequence.

We \(X\) is a Banach space, thus \(y_n\) is convergent (to some \(\tilde{y}\)) in \(X\). Strong (norm) convergence implies weak convergence and weak limits are unique, thus \(\tilde{y} = y\)

\[\|y_n - y\| \to 0\]

\[\|x_n - x\| \leq \|x_n\| \|y_n - y\| + \|y\| (\|x_n\| - \|x\|) \to 0\]

Fact: Every uniformly convex Banach space \(X\) is reflexive, i.e. \((X^*)^* = X\)
Theorem: \( L^p \) spaces are, for \( p \in (1, \infty) \), uniformly convex.

Lemma (Clarkson's inequalities)

(1) for \( p \in [2, \infty) \) and \( f, g \in L^p \)

\[
\|f + g\|_p^p + \|f - g\|_p^p \leq 2^{p-1} \left( \|f\|_p^p + \|g\|_p^p \right)
\]

(2) for \( p \in (1,2] \), \( q = \frac{p}{p-1} \) and any \( f, g \in L^p \)

\[
\|f + g\|_p^q + \|f - g\|_p^q \leq 2 \left( \|f\|_p^p + \|g\|_p^p \right)^{q-1}
\]

Proof - Exercise course

Proof of Theorem

\( p \in [2, \infty) \): \( f, g \in L^p \), \( \|f\|_p \leq 1 \), \( \|g\|_p \leq 1 \), \( \|f - g\|_p \geq \varepsilon \), then

\[
\left\| \frac{f + g}{2} \right\|_p^p = 2^{-p} \|f + g\|_p^p \leq 2^{-p} \cdot 2^{p-1} \left( \|f\|_p^p + \|g\|_p^p \right) - \|f - g\|_p^p
\]

\[
\leq 1 - \varepsilon^p
\]

Thus \( \left\| \frac{f + g}{2} \right\|_p \leq (1 - \varepsilon^p)^{1/p} \); we take \( \delta = 1 - (1 - \varepsilon^p)^{1/p} > 0 \).
\[ p \in (1, 2) \]
\[ \left\| \frac{f+g}{2} \right\|_p^q = 2^{-q} \left\| f+g \right\|_p^q \leq 2^{-q} (2 (\left\| f \right\|_p^p + \left\| g \right\|_p^p)^{q-1} - \left\| f-g \right\|_p^q) \]
\[ \leq 2^{-q} (2 \cdot 2^{q-1} - \varepsilon q) = 1 - (\frac{\varepsilon}{2})^q \]
and we can take \( \varepsilon = 1 - (1 - (\frac{\varepsilon}{2})^q)^{1/q} \).

**Lemma (McShane)**

Let \( T \) be a bounded linear functional on a normed linear space \( X \). Suppose that for some \( f, g \in X \)

\((\ast)\) \( \left\| g \right\|_p = 1 \) and \( T(g) = \left\| T \right\|_{X^*} \)

and \((\ast\ast)\) \( \lim_{t \to 0} \frac{\left\| g + \varepsilon f \right\|_p^p - \left\| g \right\|_p^p}{pt} \) exists for some \( p > 1 \).

Then \( T(f) = \left\| T \right\| \cdot \lim_{t \to 0} \frac{\left\| g + \varepsilon f \right\|_p^p - \left\| g \right\|_p^p}{pt} \).
Proof of McShane's Lemma

First, note that
\[
\lim_{t \to 0} \frac{(T(g+tf))^p - (T(g))^p}{pt} = \lim_{t \to 0} \frac{(T(g) + tT(f))^p - (T(g))^p}{pt} = T(g)^{p-1} T(f) = \|T\|^{p-1} T(f).
\]

Next, \( \|T\| \|g\| = T(g) \)
\( \|T\| \|g+tf\| \geq T(g+tf) \)

thus
\[ \lim_{t \to 0^+} \|T\|^p \frac{\|g+tf\|^p - \|g\|^p}{pt} \geq \lim_{t \to 0^+} \frac{(T(g+tf))^p - (T(g))^p}{pt} = \|T\|^{p-1} T(f) = \lim_{t \to 0^-} \frac{(T(g+tf))^p - (T(g))^p}{pt} \]

\[ \geq \|T\|^p \frac{\|g+tf\|^p - \|g\|^p}{pt} \]

By (\( \star \)), both ends are equal, thus
\[ T(f) = \|T\| \lim_{t \to 0} \frac{\|g+tf\|^p - \|g\|^p}{pt} \]
Proof of $L^p - L^q$ duality

Scheme: 1) for any $T \in (L^p)^*$ find $g \in L^p$

\[ \text{s.t. } \|g\|_p = 1, \quad T(g) = \|T\| \]

2) compute the limit from (1), McShane's Lemma

1) Choose a sequence $(g_n)$ in $L^p$ s.t.

\[ \|g_n\|_p = 1, \quad T(g_n) \rightarrow \|T\| + 0 \]

Then

\[ 2 \geq \|g_n + g_m\|_p \geq \frac{|T(g_n + g_m)|}{\|T\|} \quad n,m \rightarrow 2 \]

thus, by Lemma, $(g_n)$ is a Cauchy sequence

\[ g_n \rightarrow g \quad \Rightarrow \quad T(g_n) \rightarrow T(g) \Rightarrow T(g) = \|T\| \]

2) \[ \frac{d}{dt} \bigg|_{t=0} |a + bt|^p = p|a + bt|^{p-1} \cdot b \cdot \text{sgn}(a + bt) \bigg|_{t=0} \]

\[ = p|a|^{p-1} b \cdot \text{sgn} a \]

\[ \lim_{t \rightarrow 0} \frac{\|g + tf\|_p^p - \|g\|_p^p}{pt} = \int \lim_{t \rightarrow 0} \frac{|g + tf|_p^p - |g|_p^p}{pt} \]

Dominated Convergence Theorem
\[ \mathcal{T}(f) = \| \mathcal{T} \| \int_E \left| f \right| \cdot \left| g \right|^{p-1} \cdot \text{sgn} \cdot f \overset{\text{E}}{=} \int_E \left| f \right| \cdot \left| g \right|^{p-1} \cdot \text{sgn} \cdot f. \]

Therefore, for every \( f \in L^p \)

\[ \mathcal{T}(f) = \| \mathcal{T} \| \int_E \left| f \right| \cdot \left| g \right|^{p-1} \cdot \text{sgn} \cdot f = \mathcal{T}_h \left( f \right) \]

for \( h = \| \mathcal{T} \| \cdot \left| g \right|^{p-1} \cdot \text{sgn} \cdot g. \)