Uniformisations of regular relations over bi-infinite words

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Abstract

We consider the problem of deciding whether a given MSO-definable relation over bi-infinite words contains an MSO-definable function with the same domain. We prove that this problem is decidable. There are two obstacles to the existence of such uniformisations: the first is related to the existence of non-trivial automorphisms of bi-infinite words, whereas the second, more subtle obstacle, is related to the existence of finite, discrete dynamical systems, where no trajectory can be selected by an MSO formula.

Keywords bi-infinite words, monadic second-order logic, uniformisation

1 Introduction

This paper is concerned with the following *uniformisation problem*: does a given binary relation R admit a *uniformisation*, *i.e.* a function F with the same domain as R, whose graph is contained in R? Clearly, the axiom of choice gives a positive answer to this question, for all relations R. However, this answer is non-constructive: the obtained function is usually by all means very complicated. Constructive versions of the problem therefore seek for such uniformisations which are simple in some sense.

We consider the variant of the problem where both the relation R and the function F are required to be definable in some fixed logic, namely monadic second order logic. In this variant, the uniformisation problem can be seen as a means of assessing the expressive power of the considered logic.

Monadic second order (MSO) logic is widely studied in theoretical computer science, due to its reasonably rich expressive power and to the decidability of the satisfiability problem over various structures (words, trees, infinite words, etc.). Over sets of finite words, MSO corresponds exactly to regular languages, and this correspondence generalises to other classes of structures. Due to this, MSO-definable languages and relations are called *regular*.

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Example. Here is a simple example illustrating the regular uniformisation for finite words. Consider the relation R which consists of pairs (u, v) where $u \in \{0,1\}^*$ and $v \in \{a,b\}^*$ are such that $|u| = |v|$, and the word v has a unique position labelled b, and the corresponding position in u is labelled 1. The relation $R \subseteq \{0,1\}^* \times \{a,b\}^*$ corresponds in an obvious way to the language $L \subseteq (\{0,1\} \times \{a,b\})^*$ consisting of those words over the alphabet $\{\langle 0,a\rangle, \langle 1,a\rangle, \langle 0,b\rangle, \langle 1,b\rangle\}$, which contain exactly one letter with label $\langle 1,b\rangle$ and no letters with label $\langle 0,b\rangle$. As the language L is regular, we say that is R regular, too.

Now, a regular uniformisation F of R is a uniformisation F of R whose graph corresponds in the same way to some regular language $L' \subseteq (\{0,1\} \times \{a,b\})^*$. In this case, such a uniformisation can be easily found, by considering the language $L' \subseteq L$ of those words where the unique letter labelled $\langle 1,b\rangle$ occurs before all the letters labelled $\langle 1,a\rangle$. In fact, over finite words, every regular relation admits a regular uniformisation.

This work is based on the above meaning of a *regular relation*, for a more formal definition see Section 2.3.

This paper contributes to the study of MSO by employing the uniformisation problem as a tool for analysing the expressive power of this logic. This approach has been taken before, and a fair understanding of uniformisation for MSO over finite and infinite words and trees has been attained. Specifically, it is known that every regular relation over finite words, infinite words, and finite trees, admits a regular uniformisation, and that this property fails over infinite trees, cf. [1, 3–5, 10]. The understanding of uniformisation over infinite trees is far from complete, however: only a few examples of non-uniformisable relations are known [1, 3], and no useful characterisation of all uniformisable relations is known. In particular, even for some specific relations it is unknown whether they admit regular uniformisations [1, Conjecture 1]. In general, it seems to be a rare case that a relation involving any non-trivial guessing admits a regular uniformisation; however on the other hand, it is usually technically demanding to actually prove that no such uniformisation exists.

One way of formalising whether a characterisation is useful is to require an *effective* characterisation, *i.e.* an algorithm which inputs a regular relation and decides whether or not it admits a regular uniformisation. For infinite trees, such a characterisation seems currently beyond reach. As a step towards understanding the uniformisation problem for MSO, we consider a variant of this problem for bi-infinite words, *i.e.* words indexed by the integers $\mathbb{Z}$.

Clearly, some regular relations over such words have no regular uniformisation because one cannot even choose a unique position $x \in \mathbb{Z}$ in MSO, as the underlying structure $(\mathbb{Z}, \leqslant)$ admits automorphisms by *shifts*. One could expect that a given regular relation over bi-infinite words admits a regular uniformisation if and only if it avoids this problem with shifts. As it turns out, it is not the case: we construct an example of a regular relation which admits a uniformisation which is invariant under shifts, but does not admit any regular uniformisation. The construction is quite subtle, it involves bi-infinite words γ labelled by elements of a finite group, with an additional requirement that γ is *universal*: each finite word appears as an infix of γ infinitely many times both to the left and to the right.

Our main result provides an effective characterisation of those regular relations over bi-infinite words which admit a regular uniformisation.

A key tool in the proof is a factorisation lemma which states that there is an MSO formula that factorises any given bi-infinite word into finite factors, which are all similar to each other in some sense. Factorisation results of this type have been very important in the algebraic theory of languages of infinite words. Ramsey's theorem implies that for any infinite word w there exists an infinite set of positions X which is *homogeneous* in the sense that all the finite infixes u of w between positions in X (*i.e.* starting at a position x and ending at a position $y-1$ for some $x < y$ in X) have the same MSO *type*, *i.e.* satisfy all the same MSO formulae of fixed quantifier depth. It follows that there is a single MSO formula which in any given (one sided) infinite word w identifies a single infinite homogeneous set of positions X, namely the lexicographically smallest one. This observation easily implies that every regular relation over infinite words admits a regular uniformisation.

In the case of bi-infinite words, Ramsey's theorem still implies the existence of some infinite homogeneous set X as above in any given bi-infinite word α. However, as it turns out, there is no way of defining a single one of those homogeneous sets using an MSO formula. Our factorisation lemma states that an MSO formula can, however, define a set of positions X which is close to being homogeneous. In particular, in some cases, the infixes u of α between two positions of X might not all have the same MSO-type, but if a certain type does occur as the type of such an infix, then it also occurs as the type of an infix between two positions $x < y$ in X where x is arbitrarily far to the left and y is arbitrarily far to the right. This case turns out to be the most challenging, and gives rise to certain finite groups of types. Our characterisation of the existence of regular uniformisations is essentially tied to the analysis of finite groups which occur in this case. To prove correctness of our characterisation, we use the mentioned construction involving universal words.

Apart from the main result, we solve a number of related questions about uniformisability over bi-infinite words. For example, we characterise the existence of a regular *finitary uniformisation* (*i.e.* relation choosing a finite number of witnesses for each input), or the existence of a uniformisation which is invariant under shifts (but not necessarily definable in any logic).

Motivation. Our most important motivation is to provide the first known non-trivial algorithm deciding if a given regular relation admits any regular uniformisation. We work with bi-infinite words because we consider it the simplest case where not all regular relations have regular uniformisations. A natural next step would be to extend these results to countable (or scattered) infinite words, and then to trees with countably many branches.

Although the case of full infinite trees is much more subtle, it seems to extend the case of linear orders — the choice problem over infinite trees clearly reduces to the problem of choosing a position from a given set U that is an anti-chain w.r.t. the descendant relation. Such anti-chains are linearly ordered by the lexicographical order over the tree. Therefore, to solve the uniformisation problem over trees, understanding the case of countable linear orders seems necessary. We provide a first step in that direction.

Organisation. The paper is organised as follows. Section 2 introduces basic notions used across the paper. Section 3 provides illustrative examples of (the lack of) uniformisations over bi-infinite words. In Section 4 we state and discuss the main results of this work. Section 5 introduces the concept of *universal words*, one of the crucial technical concepts of this paper. Section 6 provides a definition and properties of algebraic structures used to recognise regular languages of bi-infinite words.

Once all these tools are introduced, we begin to gradually prove the main results. We solve the case of finitary uniformisations in Section 7. Then, in Section 8, we characterise the existence of shift-invariant uniformisations. Finally, in Section 9 we give a criterion for the existence of regular uniformisations. Section 10 shows how to put all the previous results together to obtain the general statement of Theorem 8. We conclude in Section 11.

Due to the space limitations, proofs of certain lemmas are moved to the appendix; these statements are marked by the symbol $(*)$.

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2 Preliminaries

We denote the set of natural numbers $\{0,\dots\}$ by $\mathbb{N}$ and the set of integers by $\mathbb{Z}$. A non-empty subset $X \subseteq \mathbb{Z}$ is said to be *bi-unbounded* if X has no minimal nor maximal element. An *alphabet* is any finite non-empty set of *letters*. We will use letters A and B for alphabets.

A *word* w over an alphabet A, indexed by a totally ordered set I of *positions*, is a function $w\colon I \to A$. The set of words indexed by I is denoted by A^I. *Bi-infinite words* are words indexed by $\mathbb{Z}$, *right-infinite words* (also called ω-words, or simply infinite words) are words indexed by $\{0,1,\dots\}$, *left-infinite words* are words indexed by $\{0,-1,-2,\dots\}$, and a *finite word* of length $n \geqslant 0$ is a word indexed by $\{0,1,\dots,n{-}1\}$. We denote the set of right-infinite words by A^ω, the set of left-infinite words by $A^{-\omega}$, and the set of non-empty finite words by A^+. By A^n, we denote the set of finite words of length n, and $|u|$ is the length of u. The empty word is denoted ε and $A^* = A^+ \cup \{\varepsilon\}$. A *language* is a set of words.

If $u \in A^I$ and $v \in B^I$ are two words with the same positions and over alphabets A and B, respectively, then $\langle u, v\rangle$ denotes the word over the alphabet $A \times B$ whose letter at a position $i \in I$ is the pair consisting of the ith letter of u and the ith letter of v.

Two finite words u, v can be concatenated yielding a word $u \cdot v$, a finite word v can be prepended to a right-infinite word w, yielding a right-infinite word $v \cdot w$, or appended to a left-infinite word w', yielding a left-infinite word $w' \cdot v$. Finally, a left-infinite word w' can be concatenated with a right-infinite word w, yielding a bi-infinite word $w' \cdot w$, whose ith letter is the ith letter of w for $i \geqslant 0$ and the $(i{+}1)$st letter of w' for $i < 0$. Note that $(w' \cdot v) \cdot w$ is not necessarily equal to $w' \cdot (v \cdot w)$ when v is finite, w' left-infinite, and w right-infinite. However, we will only consider the result of concatenation of left- and right-infinite words up to shifts (see below), where this discrepancy disappears.

For $x \leqslant y$ by $[x, y)$ we denote the set $V = \{x, x{+}1, \dots, y{-}1\}$ sometimes called a *factor*. If V is a factor and $\alpha \in A^{\mathbb{Z}}$ is a bi-infinite word, then by $\alpha\restriction_V \in A^*$ we denote the finite word $\alpha\restriction_V = \alpha(x)\alpha(x{+}1)\dots\alpha(y{-}1)$.

Given a finite word $u \in A^+$, denote by $u^\omega \in A^\omega$ the right-infinite word $u \cdot u \cdot u \dots$, by $u^{-\omega} \in A^{-\omega}$ the left-infinite word $\dots u \cdot u \cdot u$, and by $u^{\mathbb{Z}}$ the bi-infinite word $u^{-\omega} \cdot u^\omega$.

We will usually denote bi-infinite words using symbols α, β, γ, and δ, finite words using symbols u, v, and left- or right-infinite words using symbols w and w'.

2.1 Monadic second-order logic

A word w indexed by a totally ordered set I can be viewed as a relational structure $\widetilde{w}$ over the signature $\{\leqslant\} \cup A$, whose domain is I, with $\leqslant$ interpreted as the order on I, and where $a(i)$ holds for $i \in I$ and $a \in A$ if the ith letter of w is a.

We consider the usual semantics of monadic second-order logic (MSO), see *e.g.* [13] for a reference. Thus, it makes sense to ask if a given word α *satisfies* an MSO formula φ over the signature $\{\leqslant\} \cup A$. The *language* (of bi-infinite words) of a formula φ, denoted $\mathrm{L}(\varphi)$, is the set of bi-infinite words $\alpha \in A^{\mathbb{Z}}$ satisfying φ. Similarly, languages of infinite and finite words of φ are defined.

We say that a language is *regular* if it is a language of some MSO formula. Thus, when one needs to represent a regular language L in a computable way, it is enough to provide an MSO formula φ such that $\mathrm{L}(\varphi) = L$, and specify whether the associated language of finite, right-infinite, or bi-infinite words is considered. Similarly, when some transformation of regular languages is said to be *effective* then it means that there exists an algorithm that inputs MSO formulae representing input languages and outputs MSO formulae representing output languages.

Given an MSO formula $\varphi(x)$ with one free variable x and a word $\alpha \in A^{\mathbb{Z}}$, by $\varphi[\alpha] \subseteq \mathbb{Z}$ we denote the set of positions $x \in \mathbb{Z}$ where $\varphi(x)$ holds over α.

2.2 Shifts

Unlike finite or infinite words, bi-infinite words over an alphabet A (treated as relational structures) may have non-trivial automorphisms: *shifts*, *i.e.* automorphisms of the structure $(\mathbb{Z}, \leqslant)$. Those are exactly the bijections $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ such that $\sigma(x) = x + n$ for some fixed $n \in \mathbb{Z}$. Shifts act on bi-infinite words, treated as functions $\mathbb{Z} \to A$, in a natural way. Namely, for a shift σ and a bi-infinite word α, denote by $\sigma(\alpha)$ the bi-infinite word such that $\sigma(\alpha)(x) = \alpha(\sigma^{-1}(x))$ for $x \in \mathbb{Z}$. We say that $\alpha \in A^{\mathbb{Z}}$ is a *shift* of $\beta \in A^{\mathbb{Z}}$ if there exists a shift σ such that $\alpha = \sigma(\beta)$. This defines an equivalence relation between α and β that is denoted $\sim$. Note that $w' \cdot (u \cdot w) \sim (w' \cdot u) \cdot w$ for $u \in A^*$, $w' \in A^{-\omega}$, and $w \in A^\omega$.

A language $L \subseteq A^{\mathbb{Z}}$ is *shift-invariant* if it is a union of some $\sim$-equivalence classes. As the semantics of MSO on bi-infinite words is preserved by automorphisms of $(\mathbb{Z}, \leqslant)$, *i.e.* shifts, it follows easily that every regular language $L \subseteq A^{\mathbb{Z}}$ of bi-infinite words is shift-invariant.

We say that a bi-infinite word is *periodic* if it is a shift of the word $u^{\mathbb{Z}}$ for some $u \in A^+$. Equivalently, a bi-infinite word α is periodic if $\alpha = \sigma(\alpha)$ for some non-trivial shift σ.

Remark 1. *If $\sigma(x) = x + n$ and $\sigma(\alpha) = \alpha$ then $\alpha = u^{\mathbb{Z}}$ for some $u \in A^n$.*

2.3 Regular relations

Fix a relation $R \subseteq X \times Y$. The *domain* of R is the projection $\{x \mid (x, y) \in R\}$ of R onto X, denoted $\mathrm{dom}(R) \subseteq X$. If $\mathrm{dom}(R) = X$ then R is *total*. A relation R is *functional* (respectively *ℓ-valued*, resp. *finitely valued*, resp. *countably valued*) if for all $x \in X$, the *section* $\{y \in Y \mid (x, y) \in R\}$ has at most one element (resp. at most ℓ elements, resp. finitely many elements, resp. countably many elements). If R is a functional relation and $(x, y) \in R$, then we may say that R *maps* x to y.

In this article we are interested in relations between words and their definability in MSO. A relation $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ is *regular* if there is a regular language $L \subseteq (A \times B)^{\mathbb{Z}}$ such that a pair $(\alpha, \beta) \in A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ belongs to R if, and only if, $\langle \alpha, \beta \rangle \in L$. Similarly we define regular relations $R \subseteq A^{\omega} \times B^{\omega}$ and $R \subseteq A^* \times B^*$. Note that in the case of a regular relation $R \subseteq A^* \times B^*$, if $(u, v) \in R$ then $|u| = |v|$.

As every regular language $L \subseteq (A \times B)^{\mathbb{Z}}$ is shift-invariant, every regular relation $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ is also shift-invariant in the following sense: if $(\alpha, \beta) \in R$ then $(\sigma(\alpha), \sigma(\beta)) \in R$ for every shift σ.

2.4 Semigroups

A *semigroup* S is an algebraic structure equipped with a single binary operation $\cdot$ called the *product* that is associative, *i.e.* $s \cdot (r \cdot t) = (s \cdot r) \cdot t$. An *idempotent* is any element e of a semigroup S such that $e \cdot e = e$. When a semigroup S is known from the context, let $\sharp = |S|!$ be the factorial of the cardinality of S. Then, a standard fact of semigroup theory states that for every element $s \in S$ the power $s^{\sharp}$ is an idempotent in S.

A *group* is a semigroup G that contains a *neutral* element $1_G \in G$ (with $1_G \cdot g = g \cdot 1_G = g$ for all $g \in G$) and where each element $g \in G$ has a unique *inverse*, denoted g^{-1}, such that $g \cdot g^{-1} = 1_G = g^{-1} \cdot g$. We say that a group G is *contained* in a semigroup S (denoted simply $G \subseteq S$) if $G \subseteq S$ and the product operations $\cdot$ of G and S coincide on G.

Clearly, the set A^+ with concatenation is a semigroup (that is called the *free semigroup* over A). A language $L \subseteq A^+$ is *recognised* by a semigroup S if there is a semigroup homomorphism $h: A^+ \to S$ such that $L = h^{-1}(H)$ for some $H \subseteq S$. It is well known that $L \subseteq A^+$ is regular if and only if L is recognised by some finite semigroup. We generalize this fact to languages of bi-infinite words in Subsection 6.1.

We will sometimes consider words whose alphabet is a semigroup S. Given such a finite word $u = u_0 u_1 \ldots u_{n-1} \in A^+$ with $n = |u| > 0$, by $[\![u]\!]$ we denote the *value* of u, *i.e.* the product $u_0 \cdot u_1 \cdot \ldots \cdot u_{n-1}$ in S. If the semigroup S is known to have a neutral element 1_S then we additionally put $[\![\varepsilon]\!] = 1_S$.

2.5 Uniformisation

Assume that $F, R \subseteq X \times Y$ are two relations such that $F \subseteq R$ and $\mathrm{dom}(F) = \mathrm{dom}(R)$. We say that: F is a *uniformisation* of R if F is functional; F is an *ℓ-uniformisation* of R for $\ell > 0$ if F is ℓ-valued; F is a *finitary uniformisation* of R if F is finitely valued; and F is a *countable uniformisation* of R if F is countably valued. Clearly, every uniformisation is an ℓ-uniformisation for each ℓ; every ℓ-uniformisation is a finitary uniformisation; and every finitary uniformisation is a countable uniformisation, but the reverse implications do not hold. By the axiom of choice, every relation has (or *admits*) a uniformisation.

This paper is about the ability to find regular uniformisations of regular relations. The following two theorems show that this is always possible in the cases of finite and infinite words.

Theorem 2 (folklore). *If $R \subseteq A^* \times B^*$ is a regular relation of finite words then R effectively admits a regular uniformisation $F \subseteq R$.*

Proof sketch. It is enough to define F as the set of pairs $(u_0, v) \in A^* \times B^*$ such that $(u_0, v) \in R$ and v is the lexicographically minimal word of length $|u_0|$ such that $(u_0, v) \in R$. $\square$

Theorem 3 ([4, 8]). *If $R \subseteq A^{\omega} \times B^{\omega}$ is a regular relation of infinite words then R effectively admits a regular uniformisation $F \subseteq R$.*

Proof sketch. It is easy to observe that the above simple idea of choosing lexicographically minimal witnesses does not work in that case. As a workaround, consider a Büchi [2] automaton $\mathcal{A}$ recognising the language R. Given a word $u \in A^{\omega}$, consider the lexicographically minimal *acceptance pattern* $X \subseteq \mathbb{N}$ for u, *i.e.* the lexicographically earliest infinite set of positions where the automaton $\mathcal{A}$ can visit accepting states reading words of the form (u, v). Then, once the acceptance pattern for u is fixed, choose the lexicographically minimal run $\rho \in Q^{\omega}$ that agrees with u and has acceptance pattern X. The uniformisation F maps u to the lexicographically minimal word v that agrees with the run ρ. $\square$

There is another way of demonstrating the above result, based on *Wilke algebras*, where instead of a lexicographically minimal *acceptance pattern*, one chooses a minimal Ramsey decomposition. In this work we will use this result as a black-box, the sketch above is given just informatively.

3 Examples

In the case of bi-infinite words, some, but not all regular relations admit regular uniformisations. Consider the following examples.

Example 4. Let $R \subseteq \{a, b\}^{\mathbb{Z}} \times \{0, 1\}^{\mathbb{Z}}$ be the relation that contains a pair (α, β) if α is some shift of $a^{-\omega} \cdot b^{\omega} = \ldots aaabbb \ldots$ and β contains exactly one letter 1 (*i.e.* β is some shift of $0^{-\omega} \cdot 1 \cdot 0^{\omega} = \ldots 0001000 \ldots$). Then R admits a regular uniformisation F that maps α to the bi-infinite word β whose unique letter 1 is at the same position as the first letter b in α.

The following example shows that regular relations might not even admit shift-invariant uniformisations.

Example 5. Assume that $R \subseteq \{a\}^{\mathbb{Z}} \times \{0, 1\}^{\mathbb{Z}}$ is the relation that contains a pair (α, β) if $\alpha = a^{\mathbb{Z}}$ and β contains exactly one occurrence of the letter 1. Clearly, R is a regular relation over bi-infinite words. We show that R does not admit any uniformisation $F \subseteq R$ that is shift-invariant. In particular, R has no regular uniformisation.

Assume contrarily that $F \subseteq \{a\}^{\mathbb{Z}} \times \{0, 1\}^{\mathbb{Z}}$ is a shift-invariant uniformisation of R. Take any non-trivial shift $\sigma: \mathbb{Z} \to \mathbb{Z}$.

Let $\alpha = a^{\mathbb{Z}}$ be the unique element of $\{a\}^{\mathbb{Z}}$ and let $\beta \in \{0,1\}^{\mathbb{Z}}$ be such that $(\alpha, \beta) \in F$. Observe that $\sigma(\alpha) = \alpha$ and $\sigma(\beta) \neq \beta$ because β contains a unique position labelled 1. However, $(\alpha, \sigma(\beta))$ is a shift of (α, β) which means that both belong to F. This contradicts functionality of F.

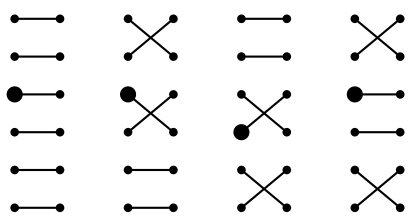
The following, more intricate example, shows a regular relation which does admit a shift-invariant uniformisation, but no regular one. This example lies at the core of this paper.

Example 6. Let $G \subseteq \Sigma_6$ be the set consisting of the following four permutations of the set $[6] \stackrel{\text{def}}{=} \{0, 1, \ldots, 5\}$ (written in cycle notation)

$$(), \quad (01)(23), \quad (23)(45), \quad (01)(45).$$

Note that G together with the operation of composition of permutations is a subgroup of the permutation group Σ_6. Define the relation $R_G \subseteq G^{\mathbb{Z}} \times [6]^{\mathbb{Z}}$ as the set of those pairs (γ, δ), where for each $x \in \mathbb{Z}$, the permutation $\gamma(x) \in G$ maps $\delta(x)$ to $\delta(x{+}1)$.

The following picture depicts graphically a sequence of the four available letters in γ and some corresponding positions chosen by δ.



Clearly, the relation R_G is regular and total (*i.e.* $\mathrm{dom}(R_G) = G^{\mathbb{Z}}$). We show that R_G admits a shift-invariant uniformisation F. Note that for the existence of F it is enough to find shift-invariant uniformisations for the restriction of R_G to each shift-equivalence class $U = [\alpha]_\sim$.

To this end, consider a shift-equivalence class $U = [\gamma]_\sim$ of some $\gamma \in G^{\mathbb{Z}}$. We show that the restriction of R_G to U (*i.e.* $R_G \cap (U \times [6]^{\mathbb{Z}})$) admits a shift-invariant uniformisation.

Consider two cases, depending on whether or not γ is periodic. If γ is not periodic, then $\sigma(\gamma) \neq \gamma$ for every non-trivial shift $\sigma\colon \mathbb{Z} \to \mathbb{Z}$. Pick an arbitrary δ such that $(\gamma, \delta) \in R_G$. Let $F \subseteq G^{\mathbb{Z}} \times [6]^{\mathbb{Z}}$ be the smallest shift-invariant relation containing (γ, δ), *i.e.* F contains pairs $(\sigma(\gamma), \sigma(\delta))$ for all shifts σ. Then $F \subseteq R_G$ since R_G is shift-invariant. Moreover, F is functional, since the mapping $\sigma \mapsto \sigma(\gamma)$ is injective by non-periodicity of γ. This defines a shift-invariant uniformisation of the restriction of R_G to U, in the case when U consists of non-periodic words.

Now, assume that $U = [\gamma]_\sim$ for some periodic word $\gamma = u^{\mathbb{Z}}$ with $u \in G^+$. Let $g = [\![u]\!] \in G$ be the product of all permutations in u. As every permutation in G has a fixpoint, there is some $p \in [6]$ such that $g(p) = p$. Fix such p. Notice that there is a unique $\delta \in [6]^{\mathbb{Z}}$ such that $(\gamma, \delta) \in R_G$ and for every $x \in \mathbb{Z}$ we have $\delta(x \cdot |u|) = p$. It is easy to see that δ is periodic and $\sigma(\delta) = \delta$ for every shift σ such that $\sigma(\gamma) = \gamma$. Hence, the relation consisting of all pairs $(\sigma(\gamma), \sigma(\delta))$, for all shifts σ, is functional. This proves that R_G has a shift-invariant uniformisation.

To show that R_G has no regular uniformisation, we construct a word $\gamma \in G^{\mathbb{Z}}$ such that there is no functional regular relation F such that $(\gamma, \delta) \in F \cap R$ for some $\delta \in [6]^{\mathbb{Z}}$. The word γ is an arbitrary *universal word*, *i.e.* a word γ such that every finite word $u \in G^{\mathbb{Z}}$ appears as an infix arbitrarily far to the right and arbitrarily far to the left in γ. It is not trivial to prove that no functional regular relation F as above exists. We do that using some algebraic lemmas in Section 5.

4 The main results

Our main result provides an effective characterisation of those regular relations which admit a regular uniformisation.

Main Theorem 1. *There exists an algorithm that inputs a regular relation $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ and answers whether R admits a regular uniformisation.*

It is not difficult to see that in Main Theorem 1, it is enough to consider total relations, as the general case easily reduces to this case, by extending a given relation $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ to the total relation $R \cup (L \times \{b^{\mathbb{Z}}\})$ where $L = A^{\mathbb{Z}} \setminus \mathrm{dom}(R)$ and $b \in B$ is fixed. The same holds for the other results, speaking about ℓ-uniformisations and finitary uniformisations.

An interesting consequence of our reasoning is the following result, which states that to uniformise a relation $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$, it is sufficient to uniformise it over each word $\alpha \in A^{\mathbb{Z}}$ separately, as formalised below. A *regular selection* [9] for a relation $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ *over* a word $\alpha \in A^{\mathbb{Z}}$ is a regular relation $F_\alpha \subseteq R$ such that there is a unique word β with $(\alpha, \beta) \in F_\alpha$. Say that R *admits regular selections* if it has a regular selection F_α over each word $\alpha \in \mathrm{dom}(R)$. Observe that a relation $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ admits a regular uniformisation if and only if the regular selection F_α may be chosen independently of α. In particular, if R admits regular uniformisation then it admits regular selections, but in general relational structures, the converse implication might fail. However, both conditions turn out to be equivalent for bi-infinite words.

Main Theorem 2. *A regular relation $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ admits a regular uniformisation if and only if it admits regular selections.*

We now sketch the idea of our characterisation of those relations R which admit regular selections. Given a word $\alpha \in A^{\mathbb{Z}}$ we distinguish two cases, depending on whether some MSO formula $\phi(x)$ can identify a single position in α, or no such formula exists. If such a formula exists, then α can be split into a left-infinite word and a right-infinite word, by cutting at the position selected by $\phi(x)$. For the two resulting words we may apply uniformisation for infinite words, given by Theorem 3, to obtain a regular selection of R over α.

The interesting case is when no MSO formula $\phi(x)$ can distinguish a single position of α. Our key technical tool, the

Factorisation Lemma below, says that then there is an MSO-definable partition of α into infinitely many finite words, called *factors* of α, which are similar to each other, in a certain sense. There are two main cases which arise here. In the first case, all the factors of α evaluate to the same element in a suitably chosen semigroup S. This case further splits into two cases: in one case it is possible to independently uniformise each factor by applying Theorem 2, obtaining a regular selection over α, and in the other case, it turns out that not even a shift-invariant selection over α exists, similarly as in Example 5.

The last case is when there is no MSO-definable partition of α into finite factors which evaluate to a single element in the semigroup S. We show that then there is a group $G \subseteq S$ such that each factor evaluates to some element of G. This case turns out to be the most interesting and technically involved. An algebraic condition on the group G allows us to identify the cases when R has a regular selection over α. When this condition fails, we are able to construct a counterexample to the existence of a regular selection by considering universal bi-infinite words over G, similarly as in Example 6.

This finishes the sketch of our effective characterisation for admitting regular selections. Main Theorem 2 can be deduced by observing that each case distinction in the sketch above can be determined by an MSO formula. This is part of the statement of the Factorisation Lemma, which we now formulate.

A *factorisation* is any bi-unbounded set $X \subseteq \mathbb{Z}$. A *factor* of a factorisation X is a set of positions $V \subseteq \mathbb{Z}$ which is an interval of the form $[x, y)$, for two positions $x, y \in X$ which are consecutive in X (*i.e.* $V \cap X = \{x\}$). If a homomorphism $h\colon A^+ \to S$ to a finite semigroup is fixed, the *value* of a factor V in α is $h(\alpha\restriction_V) \in S$. For $s \in S$, an *s-constant factorisation* is a factorisation in which all the factors have value equal to s. For a group $G \subseteq S$, a *G-group factorisation* is a factorisation in which all the factors have value in G, and which is not an s-constant factorisation for any $s \in S$. The factorisation X is usually selected by an MSO formula $\varphi(x)$ using the notation $\varphi[\alpha] = \{x \in \mathbb{Z} \mid \varphi(x) \text{ holds over } \alpha\}$.

Lemma 7 ((∗) Factorisation Lemma)**.** *Let A be a finite alphabet and $h\colon A^+ \to S$ be a homomorphism to a finite semigroup S. There effectively exists an* MSO *formula $\varphi(x)$ with the following properties. Take any word $\alpha \in A^{\mathbb{Z}}$. Then, exactly one of the following conditions holds:*

1. *$\varphi[\alpha]$ is a singleton;*
2. *$\varphi[\alpha]$ is an s-constant factorisation of α for some $s \in S$;*
3. *$\varphi[\alpha]$ is a G-group factorisation of α for some group $G \subseteq S$.*

Moreover, if S is aperiodic (contains only singleton groups), then $\varphi(x)$ is a formula of first-order logic — $\varphi(x)$ does not use second-order quantifiers.

The proof of this lemma follows the standard ways of dealing with semigroups. The crucial ingredient is given by Green's relations [7, Annex A] (a standard tool used in the algebraic theory of regular languages). The constructed formula $\varphi(x)$ inductively constructs coarser and coarser factorisations of a given word, by distinguishing positions of specific algebraic properties in S.

At each stage of the construction, either a single position is distinguished (which leads to Case 1. of the lemma), the factorisation is already fully homogeneous (all its factors have the same value in S, i.e. Case 2. holds), or the next factorisation is constructed. The consecutive factorisations aim at forcing all the factors to share the same $\mathcal{J}$-class of S. The structure of these $\mathcal{J}$-classes in S bounds the number of factorisations that are involved. Once all the factors come from the same $\mathcal{J}$-class, one can additionally force them to come from a single $\mathcal{R}$- and single $\mathcal{L}$-class. When all these steps are successfully performed, all the factors belong to a single $\mathcal{H}$-class of S. It is known that each non-trivial $\mathcal{H}$-class of S is a group, so Case 3. of the lemma holds). A complete presentation of this argument is given in Appendix E.

Using this lemma and similar techniques to the ones sketched above, we obtain effective characterisations to other questions related to uniformisations. The following result summarises those results.

Theorem 8. *There exists an algorithm that inputs a regular relation $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ and answers all the following questions, whether:*

0. *R admits a regular countable uniformisation;*
1. *R admits a regular finitary uniformisation;*
2. *R admits a regular ℓ-uniformisation for some $\ell > 0$;*
3. *R admits a shift-invariant uniformisation;*
4. *R admits a regular uniformisation;*
5. *R admits regular selections.*

Moreover, the following implications hold[1]

$$5 \Leftrightarrow 4 \Rightarrow 3 \Rightarrow 2 \Leftrightarrow 1 \Rightarrow 0 \Leftrightarrow \top$$

and there are examples of relations R such that: $4 \not\Leftarrow 3$, $3 \not\Leftarrow 2$, *and* $1 \not\Leftarrow 0$.

Finally, in the cases 0, 2, and 4, if it turns out that R admits a respective uniformisation F then the algorithm is able to construct such F.

The proofs of particular ingredients of this theorem are spread across the rest of this paper. The order in which the claims are proved is aimed at gradually building the set of tools used to demonstrate the results.

5 Universal words

To prove Main Theorem 1, we need two algebraic facts concerning universal words. They additionally imply the following claim.

[1]In particular we prove that Condition 0 is true for every regular relation R.

Claim 9. *The relation R_G described in Example 6 does not admit regular selections.*

Fix an alphabet A. A bi-infinite word $\alpha \in A^{\mathbb{Z}}$ is *universal* if for every finite word $u \in A^+$ the set of positions x such that $\alpha{\upharpoonright}_{[x,x+|u|)} = u$ is bi-unbounded. It is easy to see that universal bi-infinite words exist: an example of such a word is the following bi-infinite concatenation:

$$\ldots \cdot u_2 \cdot u_1 \cdot u_0 \cdot u_1 \cdot u_2 \cdot \ldots,$$

where $u_0, u_1, u_2, \ldots$ is any enumeration of A^+.

A *pattern* is a triple (u, a, v) with $u, v \in A^*$ and $a \in A$. We say that a pattern (u, a, v) *appears* in $\alpha \in A^{\mathbb{Z}}$ *at a position* $x \in \mathbb{Z}$ if: $\alpha(x) = a$, $\alpha{\upharpoonright}_{[x-|u|,x)} = u$, and $\alpha{\upharpoonright}_{[x+1,x+|v|+1)} = v$.

Proposition 10 ($*$). *Let $\varphi(x)$ be an* MSO *formula with one free variable x and let $\alpha \in A^{\mathbb{Z}}$ be a universal word. Assume that $\varphi(x)$ holds over α for some position $x \in \mathbb{Z}$. Then there exists a pattern (u, a, v) such that for every position $x' \in \mathbb{Z}$ if the pattern (u, a, v) appears in α at a position $x' \in \mathbb{Z}$ then $\varphi(x')$ holds over α.*

Proposition 10 follows easily from the consideration of Green's relations in an appropriate algebraic structure that corresponds to $\varphi(x)$, see Appendix C.

Lemma 11. *Let G be a finite group and $\varphi(x)$ be an* MSO *formula over the alphabet G. Let $\gamma \in G^{\mathbb{Z}}$ be a universal word and assume that $\varphi(x)$ holds in γ at some position x. Let $X = \varphi[\gamma]$ be the set of positions where $\varphi(x)$ holds over γ. Then, for every element $g \in G$ there exists a pair of positions $x < y$ both in X such that $[\![\gamma{\upharpoonright}_{[x,y)}]\!] = g$.*

Proof. Let $\gamma \in G^{\mathbb{Z}}$ be a universal word. Let $X = \varphi[\gamma]$. Apply Proposition 10 to $\varphi(x)$ obtaining a pattern (u, a, v) over the alphabet G, such that whenever this appears in γ at a position x then $x \in X$.

Consider the word obtained as the following concatenation

$$U \quad = \quad u \cdot a \cdot v \cdot ([\![v]\!]^{-1}) \cdot (a^{-1}) \cdot g \cdot ([\![u]\!]^{-1}) \cdot u \cdot a \cdot v,$$

where (a^{-1}), $([\![v]\!]^{-1})$, and $([\![u]\!]^{-1})$ are single-letter words over G.

Since γ is a universal word, there exists an occurrence of U in γ. Let $x < y$ be the positions of the two indicated letters a in that occurrence. It means that the pattern (u, a, v) appears in γ both at x and at y. Thus, both positions x and y belong to X. Moreover, we know that

$$[\![\gamma{\upharpoonright}_{[x,y)}]\!] = \left[\!\!\left[a \cdot v \cdot ([\![v]\!]^{-1}) \cdot (a^{-1}) \cdot g \cdot ([\![u]\!]^{-1}) \cdot u\right]\!\!\right] = g.$$

This concludes the proof Lemma 11. ◻

Notice that in the above construction one can insert arbitrarily many copies of the word $u \cdot a \cdot v \cdot ([\![v]\!]^{-1}) \cdot (a^{-1}) \cdot ([\![u]\!]^{-1})$ into the word U above. What means that this construction can be generalised into the following statement.

Corollary 12. *In the above lemma, one can additionally ensure that for any given n the positions x and y are chosen so that $\big|[x, y) \cap X\big| \geqslant n$.*

We can now present a proof of Claim 9 that is based on the above observations.

Proof of Claim 9. Consider the group $G \subseteq S_6$ and the relation $R_G \subseteq G^{\mathbb{Z}} \times [6]^{\mathbb{Z}}$ from Example 6. Let $\gamma \in G^{\mathbb{Z}}$ be a universal word. We show that the relation R_G does not have a regular selection over γ. In particular, R_G does not admit regular uniformisation, although it does admit a shift-invariant one, as seen in Example 6.

Assume that $F_\gamma \subseteq R_G \subseteq G^{\mathbb{Z}} \times [6]^{\mathbb{Z}}$ is a regular relation such that $(\gamma, \delta) \in F_\gamma$ for a unique $\delta \in [6]^{\mathbb{Z}}$. Let $p_0 = \delta(0) \in [6]$.

Let $\varphi(x)$ be a formula over the alphabet G which, given a word $\gamma' \in G^{\mathbb{Z}}$, states that there exists a word $\delta' \in [6]^{\mathbb{Z}}$ such that $(\gamma', \delta') \in F_\gamma$ and moreover $\delta'(x) = p_0$. Clearly, the formula $\varphi(x)$ holds in γ at a position x if and only if $\delta(x) = p_0$, in particular $\varphi(0)$ holds.

By the construction of the group G, there is some permutation $g_0 \in G$ such that $g_0(p_0) \neq p_0$. By Lemma 11 we know that there exists a pair of positions $x < y$ satisfying $\varphi(x)$ and $\varphi(y)$ over γ such that $[\![\gamma{\upharpoonright}_{[x,y)}]\!] = g_0$. By the definition of R_G we know that

$$p_0 = \delta(y) = [\![\gamma{\upharpoonright}_{[x,y)}]\!]\big(\delta(x)\big) = g_0(p_0),$$

contradicting $p_0 \neq g_0(p_0)$. This concludes the proof Claim 9. ◻

6 Algebraic preliminaries

Before moving to the rest of the proof of Theorem 8, we introduce some additional algebraic tools used in the proof.

6.1 Algebras for bi-infinite words

A *bi-Wilke algebra* (called ζ-Wilke algebra in [7]) W is an algebra with: four sorts denoted W_{fin}, $W_{\text{-}\mathit{inf}}$, W_{inf}, and W_{bi}; four operations

$$W_{\mathit{fin}} \times W_{\mathit{fin}} \to W_{\mathit{fin}} \qquad W_{\mathit{fin}} \times W_{\mathit{inf}} \to W_{\mathit{inf}}$$
$$W_{\text{-}\mathit{inf}} \times W_{\mathit{fin}} \to W_{\text{-}\mathit{inf}} \qquad W_{\text{-}\mathit{inf}} \times W_{\mathit{inf}} \to W_{\mathit{bi}}$$

all denoted $\cdot$ like concatenation; and two operations

$$W_{\mathit{fin}} \to W_{\text{-}\mathit{inf}} \qquad W_{\mathit{fin}} \to W_{\mathit{inf}}$$

denoted $s \mapsto s^{-\omega}$ and $s \mapsto s^{\omega}$, respectively. A bi-Wilke algebra needs to satisfy the following conditions:

- all operations $\cdot$ are associative, *i.e.* $(s \cdot t) \cdot r = s \cdot (t \cdot r)$ whenever s, t, r are elements of W which are compatible, *i.e.* $s \in W_{\mathit{fin}} \cup W_{\text{-}\mathit{inf}}$, $t \in W_{\mathit{fin}}$, and $r \in W_{\mathit{fin}} \cup W_{\mathit{inf}}$.
- $(s \cdot t)^{\omega} = s \cdot (t \cdot s)^{\omega}$ and $(s \cdot t)^{-\omega} = (t \cdot s)^{-\omega} \cdot t$ for $s, t \in W_{\mathit{fin}}$.
- for all $n \geqslant 1$ and $s \in W_{\mathit{fin}}$ if $r = s^n$ then $r^{\omega} = s^{\omega}$ and $r^{-\omega} = s^{-\omega}$.

Given a bi-Wilke algebra W and $s \in W_{fin}$, denote $s^{\mathbb{Z}} \stackrel{\text{def}}{=} s^{-\omega} \cdot s^{+\omega}$. Then $(s^n)^{\mathbb{Z}} = s^{\mathbb{Z}}$ for all $n \geqslant 1$.

The *free bi-Wilke algebra* generated by A, denoted F_A, has sorts A^+, $A^{-\omega}$, A^{ω}, $A^{\mathbb{Z}}/\!\sim$ consisting respectively of: all finite non-empty words, left-infinite words, right-infinite words, and bi-infinite words up to shifts, with their operations $\cdot$ and $s \mapsto s^{-\omega}$ and $s \mapsto s^{\omega}$.

Similarly as every semigroup S introduces a homomorphism $[\![.]\!]\colon S^+ \to S$ such that $[\![s]\!] = s$, we aim at defining an analogous homomorphism $[\![.]\!]\colon \mathsf{F}_{(W_{fin})} \to W$, from the free bi-Wilke algebra generated by the set W_{fin} treated as an alphabet, into W.

Definition 13. We say that a homomorphism $[\![.]\!]\colon \mathsf{F}_{(W_{fin})} \to W$ is *associative* if the following holds for every sequence $(u_x)_{x\in\mathbb{Z}}$ of words $u_x \in (W_{fin})^+$ over the alphabet W_{fin}. First, let $\gamma = \ldots \cdot u_{-1} \cdot u_0 \cdot u_1 \cdot \ldots$ be the infinite concatenation of the words $(u_x)_{x\in\mathbb{Z}}$. Now, let $\delta \in (W_{fin})^{\mathbb{Z}}$ be defined as $\delta(x) = [\![u_x]\!]$ for $x \in \mathbb{Z}$. Both γ and δ are bi-infinite words over the alphabet W_{fin}. We require that $[\![\gamma]\!] = [\![\delta]\!]$, *i.e.*

$$\Big[\!\!\Big[\ldots \cdot u_{-1} \cdot u_0 \cdot u_1 \cdot \ldots \Big]\!\!\Big] = \Big[\!\!\Big[\ldots [\![u_{-1}]\!][\![u_0]\!][\![u_1]\!] \ldots \Big]\!\!\Big]. \quad (1)$$

Definition 13 corresponds to the notion of a ζ-semigroup in [7, Section 4]. The following fact is a consequence of the discussion there (particularly, Proposition 4.2), and is a crucial property of finite bi-Wilke algebras.

Fact 14. *Every finite bi-Wilke algebra induces a unique homomorphism $[\![.]\!]\colon \mathsf{F}_{(W_{fin})} \to W$ with $[\![s]\!] = s$ for $s \in W_{fin}$ which is associative in the above sense.*

Moreover, if A is an alphabet, W a finite bi-Wilke algebra, and $h\colon A \to W_{fin}$ a function then there is a unique homomorphism (also denoted h) $h\colon \mathsf{F}_A \to W$ extending h which preserves $[\![.]\!]$ in the following sense: if $(u_x)_{x\in\mathbb{Z}}$ is a bi-infinite sequence of words in A^+ then

$$h\Big(\ldots \cdot u_{-1} \cdot u_0 \cdot u_1 \cdot \ldots \Big) = \Big[\!\!\Big[\ldots h(u_{-1})h(u_0)h(u_1) \ldots \Big]\!\!\Big], \quad (2)$$

where on the left-hand side h is applied to a bi-infinite word (yielding a value in W_{bi}); while on the right-hand side it is applied to finite words (yielding a bi-infinite sequence of values in W_{fin}).

A language $L \subseteq A^{\mathbb{Z}}$ is *recognised* by a bi-Wilke algebra W if there is an algebra homomorphism $h\colon \mathsf{F}_A \to W$ such that $L = \{\alpha \in A^{\mathbb{Z}} \mid h([\alpha]_\sim) \in H\}$ for some $H \subseteq W_{bi}$.

Notice that we use the same notation, *i.e.* $\cdot$, $(.)^{\omega}$, $(.)^{-\omega}$, and $(.)^{\mathbb{Z}}$, to denote the respective operations on words (*i.e.* in the free bi-Wilke algebra F_A) as well as in a generic bi-Wilke algebra W. However, we hope to make it clear from the context which operation is used in a particular case.

The following theorem, generalizing analogous results for finite words and for infinite words [14], relates recognisability by bi-Wilke algebras to regularity (cf. [7, Theorem 4.3 and Theorem 7.1]).

Theorem 15. *A language L of bi-infinite words is regular if and only if it is recognised by a finite bi-Wilke algebra. Moreover, the translations between* MSO *formulae and homomorphisms are effective.*

6.2 The powerset semigroup

When trying to uniformise a relation $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$, one of the crucial goals is to understand the algebraic structure of the operation of projection of R onto the alphabet A. Such an operation corresponds to the powerset operation on the respective algebras. As it turns out, in this paper it is enough to introduce it for semigroups — structures suited for finite words.

Let S be a semigroup with a product operation $\cdot$. The *powerset semigroup*, denoted $\wp(S)$, is the semigroup whose elements are all subsets of S, and the product satisfies $X \cdot Y = \{x \cdot y \mid x \in X, y \in Y\}$ for $X, Y \subseteq S$. For a semigroup homomorphism $h\colon (A{\times}B)^+ \to S$, let $\wp(h)\colon A^+ \to \wp(S)$ denote the homomorphism such that $\wp(h)(u) = \{h(\langle u, v\rangle) \mid v \in A^+, |v| = |u|\}$.

Fact 16. *If $h\colon (A \times B)^+ \to S$ is a homomorphism recognising a language $L \subseteq (A \times B)^+$ then $\wp(h)\colon A^+ \to \wp(S)$ recognises $\{u \in A^+ \mid \langle u, v\rangle \in L\}$.*

Observe that even if a homomorphism $h\colon (A \times B)^+ \to S$ is onto (*i.e.* $h\big((A \times B)^+\big) = S$) then $\wp(h)$ is in general not onto $\wp(S)$. Therefore, we will denote by $\wp_h(S) \subseteq \wp(S)$ the range of $\wp(h)$.

7 Regular finitary uniformisations

This section is devoted to an effective characterisation of Conditions 1 and 2 of Theorem 8. In particular, we characterise those regular relations which admit a finitary uniformisation.

The goal of this section is to prove the following proposition, yielding an effective characterisation of the existence of finitary uniformisations.

Proposition 17. *Given a total regular relation $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ recognised by a homomorphism h into a bi-Wilke algebra W with an accepting set $H \subseteq W_{bi}$, the following conditions are equivalent:*

1. *R admits a regular finitary uniformisation;*
2. *R admits a regular ℓ-uniformisation for some $\ell > 0$;*
3. *R admits a regular ℓ_0-uniformisation for $\ell_0 = |W_{fin}|!$;*
4. *for every idempotent $E \in \wp_h(W_{fin})$ there exists a* witness *$e \in E$ such that $e^{\mathbb{Z}} \in H$.*

Moreover, the last condition can be effectively checked based on the representation of R.

The implications $3 \Rightarrow 2 \Rightarrow 1$ are clear. The implication $1 \Rightarrow 4$ is relatively easy, its proof is given in Subsection 7.1 below. The rest of this section is devoted to the only remaining implication $4 \Rightarrow 3$.

7.1 Proof of the implication $1 \Rightarrow 4$

Consider a total regular relation $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ recognised by a homomorphism h into a bi-Wilke algebra W with an accepting set $H \subseteq W_{bi}$. We need to show that if R admits a regular finitary uniformisation $F \subseteq R$ then for every idempotent $E \in \wp_h(W_{fin})$ there exists a *witness* $e \in E$ such that $e^{\mathbb{Z}} \in H$.

Consider an idempotent $E \in \wp_h(W_{fin})$ and let $u \in A^+$ be a word such that $\wp(h)(u) = E$ (we use the fact that $\wp_h(W_{fin})$ is defined as the range of $\wp(h)$). Let $\alpha = u^{\mathbb{Z}}$ and take $\beta \in B^{\mathbb{Z}}$ such that $(\alpha, \beta) \in F \subseteq R$ (in particular $h(\langle \alpha, \beta \rangle) \in H$).

Claim 18. *The word β is periodic, i.e. $\beta = v^{\mathbb{Z}}$.*

Proof. If β is not periodic, then for every two distinct shifts $\sigma_1 \neq \sigma_2$, we have $\sigma_1(\beta) \neq \sigma_2(\beta)$. Consider the infinite family of shifts $\sigma_i(x) = x + i \cdot |u|$. Then, for every $i \neq j \in \mathbb{Z}$ we have $\sigma_i(\alpha) = \sigma_j(\alpha) = \alpha$ and $\sigma_i(\beta) \neq \sigma_j(\beta)$. Therefore, $\{(\alpha, \sigma_i(\beta)) \mid i \in \mathbb{Z}\} \subseteq F$ is an infinite set, contradicting the fact that F is a finitary uniformisation. □

Let $\beta = v^{\mathbb{Z}}$. Then, α and β can be written as $\left(u^{|v|}\right)^{\mathbb{Z}}$ and $\left(v^{|u|}\right)^{\mathbb{Z}}$ respectively. Observe that $\wp(h)(u^{|v|}) = \wp(h)(u) = E$, because E is an idempotent. Take e as $h(\langle u^{|v|}, v^{|u|} \rangle)$. Clearly, $e \in E$. Moreover,

$$e^{\mathbb{Z}} = h\left(\langle u^{|v|}, v^{|u|} \rangle^{\mathbb{Z}}\right) = h\left(\langle \alpha, \beta \rangle\right) \in H.$$

7.2 Case-study in the implication $4 \Rightarrow 3$

We now move to the proof of the implication $4 \Rightarrow 3$. Apply Lemma 7 to the homomorphism

$$\wp(h) \colon A^+ \to \wp_h(W_{fin})$$

yielding a formula $\varphi(x)$. Let $D_0 \subseteq A^{\mathbb{Z}}$ be the set of words $\alpha \in A^{\mathbb{Z}}$ where $\varphi(x)$ defines a single position. Similarly, let $D_K \subseteq A^{\mathbb{Z}}$ be the set of words where $\varphi(x)$ defines a K-constant factorisation for $K \in \wp_h(W_{fin})$ and $D_G \subseteq A^{\mathbb{Z}}$ be the set of words where $\varphi(x)$ defines a G-group factorisation for a group $G \subseteq \wp_h(W_{fin})$. All the above languages are regular so it is enough to find separately ℓ_0-uniformisations for the finite family of relations $R_0 = R \cap \left(D_0 \times B^{\mathbb{Z}}\right)$, $R_K = R \cap \left(D_K \times B^{\mathbb{Z}}\right)$, and $R_G = R \cap \left(D_G \times B^{\mathbb{Z}}\right)$ for $K \in \wp_h(W_{fin})$ and $G \subseteq \wp_h(W_{fin})$ a group. This is done in the following three subsections.

7.3 Single position case

The case of R_0 follows easily from Theorem 3, as $\phi(x)$ splits each word $\alpha \in D_0$ into two infinite words. This is stated in the following lemma, which is proved in Appendix D.

Lemma 19 ($*$)**.** *Let $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ be a regular relation and let $\phi(x)$ be an* MSO *formula such that $\phi(x)$ defines a single position in each $\alpha \in$ dom(R). Then the relation R admits a regular uniformisation.*

Remark 20. *The above fact implies that every regular relation admits a countable uniformisation: guess any position $x \in \mathbb{Z}$ and apply the uniformisation formula from Lemma 19. This proves that Condition 0 of Theorem 8 always holds.*

7.4 Constant factorisation case

We will now show how to define an ℓ_0-uniformisation F_K of R_K for $K \in \wp_h(W_{fin})$. Let $E = K^{\ell_0} = K^{\sharp}$ be the idempotent power of K. Pick a witness $e \in E$ such that $e^{\mathbb{Z}} \in H$ and let $F_e \subseteq A^+ \times B^+$ be a regular uniformisation[2] of the relation corresponding to the language $h^{-1}(e) \subseteq (A \times B)^+$. Let F_K contain all pairs (α, β) satisfying the following conditions. First, $\alpha \in D_K$. Let X be its K-constant factorisation, i.e. the set of positions where $\varphi(x)$ holds. Now, there must exist a set $X' \subseteq X$ that contains every ℓ_0th consecutive position of X — X' must be a factorisation and for every factor V of X' it must hold that $|V \cap X| = \ell_0$. Moreover, $(\alpha{\restriction}_V, \beta{\restriction}_V) \in F_e$, for every factor V of X'. Intuitively, the relation F_K groups the consecutive ℓ_0 factors of X and uniformises them independently, using F_e.

Notice that the relation F_K is MSO-definable. Moreover, all the choices in the definition are uniquely defined based on α, except for $X' \subseteq X$ which can be chosen in exactly ℓ_0 different ways. We show that F_K is ℓ_0-valued, by showing that for $\alpha \in$ dom(R_K) and $X' \subseteq X$ as above, there is a unique $\beta \in B^{\mathbb{Z}}$ such that $(\alpha{\restriction}_V, \beta{\restriction}_V) \in F_e$ for every factor V of X', and moreover, $(\alpha, \beta) \in R$.

Observe that each factor V of the factorisation X' as above is a union of exactly ℓ_0 factors of the factorisation X. Since $K^{\ell_0} = E$, each such factor V has value E, *i.e.* $\wp(h)(\alpha{\restriction}_V) = E$. Since $e \in E$, by definition of $\wp(h)$, there exists a word $v \in B^{|V|}$ such that $h(\langle \alpha{\restriction}_V, v \rangle) = e$. As F_e is a uniformisation of the relation corresponding to $h^{-1}(e)$, there is a unique word v_0 such that $\langle \alpha{\restriction}_V, v \rangle_0 \in F_e$. This proves uniqueness of β, given α and X'. Moreover, by the above argument, $h(\langle \alpha, \beta \rangle{\restriction}_V) = e$ for each factor V of X', and hence $h(\langle \alpha, \beta \rangle) = e^{\mathbb{Z}} \in H$, proving that $(\alpha, \beta) \in R$.

7.5 Group factorisation case

What remains is to define an ℓ_0-uniformisation F_G of R_G for a group $G \subseteq \wp_h(W_{fin})$. The idea of the construction is similar to the case of R_K for $K \in \wp_h(W_{fin})$, however, there are some additional technical difficulties in that case.

The general idea is to define, given $\alpha \in$ dom(R_G), a factorisation of α into factors of value E (the identity of G) and uniformise each of them separately, similarly as in the K-constant case above. However, this is not always possible, as illustrated in the following example.

Example 21. Consider G to be the group with elements $\{-1, 1\}$ and multiplication. Let $\alpha \in G^{\mathbb{Z}}$ be the word such that $\alpha(0) = -1$ and $\alpha(x) = 1$ for $x \neq 0$. Then there is no factorisation of α where each of the factors has value $1 \in G$.

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[2]Such F_e exists thanks to Theorem 2.

We therefore aim to distinguish between the case as above, where the word can be split using a fixed MSO formula, and the remaining case.

Consider a word $\alpha \in D_G$ and its G-group factorisation X. Given two positions $x, y \in X$, define $g_{x,y} \in G$ as the value $h(\alpha\restriction_{[x,y)})$ for $y > x$, as $(g_{y,x})^{-1}$ for $y < x$, and as the identity $E \in G$ for $x = y$. Then $g_{x,y} \in G$ for all $x, y \in X$ since X is a G-group factorisation. Moreover, for all $x, y, z \in X$ we have

$$g_{x,z} = g_{x,y} \cdot g_{y,z}. \tag{3}$$

For a position $x \in X$, denote by $\gamma_x \in G^X$ the word, indexed by X, satisfying $\gamma_x(z) = g_{x,z}$ for all $z \in X$. It follows from (3) that $\gamma_x(z) = g_{x,y} \cdot \gamma_y(z)$ for any two positions $x, y \in X$. In other words, the word γ_x is obtained from the word γ_y by multiplying pointwise by the element $g_{x,y} \in G$ from the right. Notice that there is at most $|G|$ words of the form γ_x, for $x \in X$, i.e. $\big|\{\gamma_x \mid x \in X\}\big| < |G|$.

Say that the word α is *homogeneous* if for each $g \in \mathrm{Im}(\gamma_x)$ that appears in γ_x, the set $\gamma_x^{-1}(g)$ of occurrences of g in γ_x is bi-unbounded. It follows from the above discussion that this notion does not depend on the choice of the position of $x \in \mathbb{Z}$. Moreover, it is not difficult to see that there is an MSO formula which determines whether α is homogeneous.

Let $D_+ \subseteq D_G$ denote the set of homogeneous words, and let D_- be the set of non-homogeneous words in D_G. Let $R_+ = R_G \cap (D_+ \times B^{\mathbb{Z}})$ and $R_- = R_G \cap (D_- \times B^{\mathbb{Z}})$ be the restrictions of R_G to D_+ and D_-, respectively. Since the set of homogeneous words is MSO-definable, the relations R_+ and R_- are regular. We will define an ℓ_0-uniformisation F_+ of R_+ and a uniformisation F_- of R_-. Then $F_+ \cup F_-$ will be an ℓ_0-uniformisation of R_G.

Homogeneous subcase. Fix $\alpha \in D_+$, a position $x \in X$, and a letter $g \in \mathrm{Im}(\gamma_x)$.

Claim 22. *The set $\gamma_x^{-1}(g)$ is a factorisation of α where each factor has value E.*

Proof. The set $\gamma_x^{-1}(g)$ is bi-unbounded by homogeneity of α. Let $z, z' \in \gamma_x^{-1}(g)$, so that $\gamma_x(z) = \gamma_x(z') = g$. In particular, $g_{x,z} = g_{x,z'}$, implying that $g_{z,z'}$ is the neutral element $E \in G$ because of (3). This yields the conclusion. □

Let F_+ contain all the pairs (α, β) such that $\alpha \in D_+$ and there is some $x \in \mathbb{Z}$ and $g \in \mathrm{Im}(\gamma_x)$ such that $(\alpha\restriction_V, \beta\restriction_V) \in F_e$ for every factor V of the factorisation $\gamma_x^{-1}(g)$. Similarly as in the case of F_K, the above definition provides a $|G|$-uniformisation of the relation R_+, since the possible choices of x and g lead only to at most $|G|$ distinct factorisations $\gamma_x^{-1}(g)$ of α.

Non-homogeneous subcase. Let $\alpha \in D_-$. Then, for each $x \in X$ there is some g such that $\gamma_x^{-1}(g)$ is non-empty and not bi-unbounded, *i.e.* it either contains a largest position $z \in X$, or a smallest position $z \in X$. Call such a position $z \in X$ *extreme* for x. In particular, x has at most $|G|$ extreme positions, and every other position $y \in X$ has exactly the same set of extreme positions, since γ_x and γ_y are related by multiplication by some $g \in G$. In particular, the largest extreme position x_0 for x does not depend on the choice of x. In this case, we may use x_0 to split the word α. More precisely, there is an MSO formula $\psi(x_0)$ which holds in $\alpha \in D_-$ at a position $x_0 \in \mathbb{Z}$ if and only if x_0 is the largest extreme position for some $x \in X$. Applying Lemma 19 to $\psi(x)$ and R_- yields a regular uniformisation F_- of R_-.

Remark 23. *The construction presented above that distinguishes the homogeneous and non-homogeneous cases works for an arbitrary semigroup S. This means that the statement of Lemma 7 can be strengthened by requiring that in the case of a G-group factorisation X, this factorisation has additional property that the factors can be grouped further (in one of at most $|G|$ possible ways) into a coarser factorisation with values constantly equal 1_G.*

8 Shift-invariant uniformisations

The next stage of the proof of Theorem 8 is to provide an effective characterisation of the existence of (possibly non-regular) shift-invariant uniformisations. Our characterisation is expressed by the following proposition.

Proposition 24. *Let $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ be a total shift-invariant relation. The following conditions are equivalent:*

1. *R admits a uniformisation that is shift-invariant;*
2. *for every $\alpha \in A^{\mathbb{Z}}$ there exists a shift-invariant uniformisation of the restriction $R \cap \big([\alpha]_\sim \times B^{\mathbb{Z}}\big)$, where $[\alpha]_\sim = \{\sigma(\alpha) \mid \sigma \text{ is a shift}\}$;*
3. *for every $u \in A^+$ there exists $v \in B^+$ with $|v| = |u|$ such that $(u^{\mathbb{Z}}, v^{\mathbb{Z}}) \in R$.*

Moreover, if the relation R is recognised by a homomorphism h into a bi-Wilke algebra W with an accepting set $H \subseteq W_{bi}$ then the above conditions are equivalent to:

4. *for every $K \in \wp_h(W_{fin})$ there is some $k \in K$ such that $k^{\mathbb{Z}} \in H$.*

The implication 2 ⇒ 1 has been observed in Example 6. The implication 3 ⇒ 2 is proved in the same way as done in a special case in Example 6, we present here a proof for the sake of completeness.

Proof of 3 ⇒ 2. Take any word $\alpha \in A^{\mathbb{Z}}$. First consider the case that α is not periodic. Choose any word β such that $(\alpha, \beta) \in R$. Let $F_\alpha \stackrel{\text{def}}{=} \{(\sigma(\alpha), \sigma(\beta)) \mid \sigma \text{ is a shift}\}$. Since α is not periodic, $\sigma(\alpha) \neq \alpha$ for every non-trivial shift σ. Therefore, the relation F_α is functional. Clearly, $\mathrm{dom}(F_\alpha) = [\alpha]_\sim$ and F_α is shift-invariant. Finally, since $(\alpha, \beta) \in R$ and R is shift-invariant, we know that $F_\alpha \subseteq R$. Thus, F_α is a shift-invariant uniformisation of $R \cap \big([\alpha]_\sim \times B^{\mathbb{Z}}\big)$.

Now consider the case that α is periodic, *i.e.* $\alpha = u^{\mathbb{Z}}$ for some $u \in A^+$. Let $v \in A^+$ be given by Condition 3. Let $F_\alpha \stackrel{\text{def}}{=}$

$\{(\sigma(u^{\mathbb{Z}}), \sigma(v^{\mathbb{Z}})) \mid \sigma$ is a shift$\}$. Again, $\mathrm{dom}(F_\alpha) = [\alpha]_\sim$ and as $|u| = |v|$, we know that F_α is functional: if $\sigma(u^{\mathbb{Z}}) = u^{\mathbb{Z}}$ then $\sigma(v^{\mathbb{Z}})$ is also $v^{\mathbb{Z}}$. Moreover, as $(u^{\mathbb{Z}}, v^{\mathbb{Z}}) \in R$ and R is shift-invariant, we know that $F_\alpha \subseteq R$. Clearly F_α is shift-invariant by the definition, so it satisfies the requirement of Condition 2. □

We argue that $1 \Rightarrow 3$. Fix a relation $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ and its shift-invariant uniformisation F. Take any word $u \in A^+$ and let $\alpha = u^{\mathbb{Z}}$ be its bi-infinite repetition. Let α be mapped to β by F. Let σ be the shift $x \mapsto x + |u|$, so that $\sigma(u^{\mathbb{Z}}) = u^{\mathbb{Z}}$. Since F is shift-invariant, F maps $\alpha = \sigma(\alpha)$ to $\sigma(\beta)$ and since F is functional $\sigma(\beta) = \beta$. Therefore, β can be written as $v^{\mathbb{Z}}$ for some $v \in B^+$ with $|v| = |u|$, see Remark 1. This proves Condition 3.

The equivalence between Conditions 3 and 4 is immediate from the definition of $\wp_h(W_{fin})$.

9 Regular uniformisations

We finally move to the crucial characterisation of the existence of a regular uniformisation of a given regular relation.

Proposition 25. *Let $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ be a total regular relation recognised by a homomorphism h into a bi-Wilke algebra W with an accepting set $H \subseteq W_{bi}$. The following conditions are equivalent:*

1. *R admits a regular uniformisation;*
2. *R admits a regular selection: for every word $\alpha \in A^{\mathbb{Z}}$ there exists a regular relation $F_\alpha \subseteq R$ such that there is a unique word β satisfying $(\alpha, \beta) \in F_\alpha$;*
3. *the following two conditions are satisfied:*
   - *for every $K \in \wp_h(W_{fin})$ there is some $k \in K$ such that $k^{\mathbb{Z}} \in H$,*
   - *for every group $G \subseteq \wp_h(W_{fin})$ there is a group $\Gamma \subseteq W_{fin}$ such that for all $g \in G$, the set $g \cap \Gamma$ is non-empty and $(1_\Gamma)^{\mathbb{Z}} \in H$.*

The rest of this section is devoted to a proof of this theorem. The implication $1 \Rightarrow 2$ is obvious. The following two subsections demonstrate the implications $3 \Rightarrow 1$ and $2 \Rightarrow 3$.

9.1 From criterion to uniformisation

In this section we assume that Condition 3 is satisfied and construct a regular uniformisation of R.

As in the proof of Proposition 17, we will use Lemma 7 for $\wp_h(W_{fin})$ to obtain a formula $\varphi(x)$, and separately uniformise the parts of the relation R that correspond to words α over which $\varphi(x)$ defines: a single position (denoted R_0); a K-constant factorisation for $K \in \wp_h(W_{fin})$ (denoted R_K), and a G-group factorisation for a group $G \subseteq \wp_h(W_{fin})$ (denoted R_G).

The case of R_0 follows from Lemma 19. We treat the cases of R_K and R_G below.

9.1.1 Constant factorisation case

For words where $\varphi(x)$ defines a K-constant factorisation, the uniformisation F_K works as follows. Let $k \in K$ be as in Condition 3, so that $k^{\mathbb{Z}} \in H$. Pick a uniformisation $F_k \subseteq A^+ \times B^+$ of the relation corresponding to the language $h^{-1}(k) \subseteq (A \times B)^+$. Define F_K as the set of all pairs (α, β) such that $\alpha \in \mathrm{dom}(R_K)$ and for each factor V of the factorisation X defined by $\phi(x)$ on α, the pair $(\alpha\restriction_V, \beta\restriction_V)$ belongs to F_k. As in the proof of Proposition 17 (cf. Subsection 7.4), the relation F_K is a uniformisation of R_K.

9.1.2 Group factorisation case

Now consider the case of words over which $\varphi(x)$ defines a G-group factorisation for a group $G \subseteq \wp_h(W_{fin})$. Let $\Gamma \subseteq W_{fin}$ be as in Condition 3. For each element $g \in G$ fix a witness $c_g \in g \cap \Gamma$, and let F_g be a uniformisation of the relation corresponding to the language $h^{-1}(c_g) \subseteq (A \times B)^+$. Let F_G consist of all pairs (α, β) such that $\alpha \in \mathrm{dom}(R_G)$ and for each factor V of the factorisation X defined by $\phi(x)$ in α, if $g = \wp(h)(\alpha\restriction_V) \in G$ then $(\alpha\restriction_V, \beta\restriction_V) \in F_g$.

As previously, it is easy to see that F_G is functional, because the choice of X is defined uniquely and the relations F_g are functional. Moreover, F_G is MSO-definable.

We argue that $F_G \subseteq R$. Take any pair $(\alpha, \beta) \in F_G$, and let X be the factorisation of α defined by $\phi(x)$. It follows from the definition of $\wp(h)$ that X is also a factorisation of $\langle \alpha, \beta \rangle$ such that the value of each of the factors with respect to h belongs to Γ. Let $\gamma \in \Gamma^X$ be the X-indexed word such that for $x \in X$ the letter $\gamma(x) \in \Gamma$ is the value of the factor V_x containing x with respect to h, *i.e.* $\gamma(x) = h\big(\langle \alpha\restriction_{V_x}, \beta\restriction_{V_x} \rangle\big)$. Then $[\![\gamma]\!] = h(\langle \alpha, \beta \rangle)$ because of (2) in Fact 14. Notice that formally γ is an X-indexed word, however when computing the value $[\![\gamma]\!]$ we can reindex it into a bi-infinite word.

Now we invoke the following algebraic lemma to show that $[\![\gamma]\!] = (1_\Gamma)^{\mathbb{Z}} \in H$ and therefore $(\alpha, \beta) \in R$. It is a simple extension of an analogous fact for Wilke-algebras, we provide a proof of this fact for the sake of completeness.

Lemma 26. *Fix a bi-Wilke algebra W and group $G \subseteq W_{fin}$. Let $\gamma \in G^{\mathbb{Z}}$ be any word over G. Then the product $[\![\gamma]\!]$ of the word γ in W equals $(1_G)^{\mathbb{Z}}$.*

Proof. Let $G = \{g_1, g_2, \ldots, g_n\}$ be an enumeration of all the elements of G. Fix a word $u = g_1(g_1)^{-1} g_2(g_2)^{-1} \ldots g_n(g_n)^{-1}$ of length $2n$. Notice that $[\![u]\!] = 1_G$. Construct a new word $\delta \in G^{\mathbb{Z}}$ obtained from γ by putting u in-between every two letters:

$$\delta = \ldots \cdot u \cdot \gamma(-1) \cdot u \cdot \gamma(0) \cdot u \cdot \gamma(1) \cdot u \cdot \ldots,$$

i.e. a bi-infinite concatenation of the word u with single-letter words $\gamma(x)$.

Clearly, there exists a factorisation of δ that has factors $(V_x)_{x \in \mathbb{Z}}$ of size $2n+1$ such that $\delta\restriction_{V_x} = u \cdot \gamma(x)$. Therefore, $[\![\delta\restriction_{V_x}]\!] = \gamma(x)$ and by (1) from Definition 13 we know that $[\![\delta]\!] = [\![\gamma]\!]$.

However, u has the property that for every $g \in G$ there exists a prefix v of u such that $[\![v]\!] = g$. The same holds for suffixes of u. This allows us to inductively construct another factorisation Y of δ with factors $(U_x)_{x\in\mathbb{Z}}$ such that $[\![\delta\restriction_{U_x}]\!] = 1_G$ for each $x \in \mathbb{Z}$. To obtain such a factorisation one can start in any position of δ and proceed inductively in both directions. Therefore, Equation (1) implies that $[\![\delta]\!] = (1_G)^{\mathbb{Z}}$. This concludes the proof of Lemma 26. □

At this moment we know that F_G is functional and $F_G \subseteq R$. The fact that $\mathrm{dom}(F_G) = \mathrm{dom}(R_G)$ is proved analogously as in the proof of Proposition 17 — given a word $\alpha \in \mathrm{dom}(R_G)$, one can use the uniformisations F_g to define a word β separately for each factor V of the factorisation, in such a way that $(\alpha\restriction_V, \beta\restriction_V) \in F_g$. Therefore, F_G is a regular uniformisation of R_G.

This concludes the proof of the implication 3 ⇒ 1 of Proposition 25.

9.2 From selection to the criterion

We now prove the implication 2 ⇒ 3 of Proposition 25: we assume that $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ admits a regular selection and prove the two items of Condition 3 there.

Since every regular language is shift-invariant, we may apply the implication 2 ⇒ 4 from Proposition 24, proving the first part of Condition 3 in Proposition 25. Therefore, it remains to prove the second part of that condition.

Consider a group $G \subseteq \wp_h(W_{fin})$. Let $\gamma \in G^{\mathbb{Z}}$ be a universal word over G. For each element $g \in G$ fix a word $u_g \in A^+$ such that $\wp(h)(u_g) = g$. We construct a word α as the concatenation of the words u_g in the order given by γ:

$$\alpha \stackrel{\text{def}}{=} \dots \cdot u_{\gamma(-1)} \cdot u_{\gamma(0)} \cdot u_{\gamma(1)} \cdot \dots \tag{4}$$

Fix a regular relation $F_\alpha \subseteq R$ that performs selection over α in R. Let β be the unique word such that $(\alpha, \beta) \in F_\alpha$. The word β can be split into factors $(v_x)_{x\in\mathbb{Z}}$ that correspond to the factors in (4), so that

$$\langle \alpha, \beta\rangle = \dots \cdot \langle u_{\gamma(-1)}, v_{-1}\rangle \cdot \langle u_{\gamma(0)}, v_0\rangle \cdot \langle u_{\gamma(1)}, v_1\rangle \cdot \dots$$

This allows us to define a word $\delta \in W_{fin}^{\mathbb{Z}}$ with $\delta(x)$ equal to $h\big(\langle u_{\gamma(x)}, v_x\rangle\big)$.

For each $x \in \mathbb{Z}$ we have $\delta(x) \in \gamma(x)$, because of the choice of u_g. Moreover, since $(\alpha, \beta) \in R$, we know that $h\big(\langle \alpha, \beta\rangle\big) \in H$. Equation (2) in Fact 14 implies that $h\big(\langle \alpha, \beta\rangle\big) = [\![\delta]\!]$, hence $[\![\delta]\!] \in H$.

Fact 27. *We know that F_α is a fixed regular relation that uniquely determines β given α. Moreover, the lengths of the words $u_{\gamma(x)}$ and the corresponding words v_x are bounded (there is finitely many such words). Thus, one can interpret the word δ in* MSO *over γ.*

More formally, there is a family of MSO *formulae $\lambda_s(x)$ for $s \in W_{fin}$ such that γ satisfies $\lambda_s(x)$ at a position $x \in \mathbb{Z}$ if and only if $\delta(x) = s$.*

Let $\varphi(x)$ be the formula given by Lemma 7 for the identity homomorphism id: $W_{fin} \to W_{fin}$ that chooses certain positions of the word $\delta \subseteq (W_{fin})^{\mathbb{Z}}$. The above fact allows us to interpret the formula $\varphi(x)$ over the word γ. More formally, there exists a formula $\varphi'(x)$ that holds over γ at a position $x \in \mathbb{Z}$ if and only if the formula $\varphi(x)$ from Lemma 7 holds over δ at the position x.

We now study the three possible cases given by Lemma 7. Firstly, if $\varphi(x)$ defines a single position on δ then $\varphi'(x)$ holds for a single position of γ. This is a contradiction with Lemma 11 because γ is a universal word.

9.2.1 Constant factorisation case

The second case is that $\varphi(x)$ gives a k-constant factorisation X of the word δ for some $k \in W_{fin}$. In that case, Equation (1) from Definition 13 implies that $[\![\delta]\!] = k^{\mathbb{Z}}$ and hence $k^{\mathbb{Z}} \in H$.

Recall that $k^\sharp$ is the idempotent power of $k \in W_{fin}$. Let Γ be the subgroup of W_{fin} generated by $k^{\sharp+1}$, *i.e.* $\Gamma = \{k^\sharp, k^{\sharp+1}, \dots\}$. Clearly, $(1_\Gamma)^{\mathbb{Z}} = (k^\sharp)^{\mathbb{Z}} = k^{\mathbb{Z}} \in H$. Therefore, it remains to show that for each $g \in G$ we have $g \cap \Gamma \neq \varnothing$.

The following claim follows directly from Corollary 12.

Claim 28. *There exist two positions $x < y$ in X such that $\varphi'(x)$ and $\varphi'(y)$ hold over γ, there are at least $\sharp$ positions in $[x, y) \cap X$, and $[\![\gamma\restriction_{[x,y)}]\!] = g$.*

Fix the two positions x and y above. As each factor V of X satisfies $[\![\delta\restriction_V]\!] = k$, we know that $[\![\delta\restriction_{[x,y)}]\!] = k^n$ for n such that $n = \big|[x, y) \cap X\big| \geqslant \sharp$. Therefore, $[\![\delta\restriction_{[x,y)}]\!] \in \Gamma$. On the other hand, $[\![\gamma\restriction_{[x,y)}]\!] = g$. The local condition $\delta(z) \in \gamma(z)$ clearly extends to factors, which means that $k^n = [\![\delta\restriction_{[x,y)}]\!] \in [\![\gamma\restriction_{[x,y)}]\!] = g$. Therefore, we know that $g \cap \Gamma \neq \varnothing$. This concludes the proof in that case.

9.2.2 Group factorisation case

Finally consider the case that X is a Γ-group factorisation for some group $\Gamma \subseteq W_{fin}$. By invoking Lemma 11 we argue that each $g \in G$ appears as a product of some consecutive factors of the factorisation X on γ, *i.e.* $g = [\![\gamma\restriction_{[x,y)}]\!]$ for some $x < y$ both in X. This means that $g \cap \Gamma$ is non-empty, because it contains $[\![\delta\restriction_{[x,y)}]\!]$. By invoking Lemma 26 we know that $(1_\Gamma)^{\mathbb{Z}}$ equals the product $[\![\delta]\!]$ of the types in δ. Since $[\![\delta]\!] \in H$, we know that that $(1_\Gamma)^{\mathbb{Z}} \in H$, which concludes the proof of Proposition 25.

10 Summary of the proof of Theorem 8

First, the implications 4 ⇒ 3 and 2 ⇒ 1 ⇒ 0 are obvious. The implication 3 ⇒ 2 follows from Propositions 24 and 17, because the effective conditions there are the same, except that in Proposition 17 we only care about idempotents. The equivalence 5 ⇔ 4 is a consequence of Proposition 25 and the equivalence 2 ⇔ 1 is a consequence of Proposition 17. Finally, the fact that Condition 0 always holds follows from

Remark 20. Also, the above mentioned results show effectiveness of the respective conditions.

The fact that $4 \nLeftarrow 3$ follows from Example 6 and Claim 9. The fact that $3 \nLeftarrow 2$ follows from the following example.

Example 29. Let $R \subseteq \{a\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}}$ contain a pair (α, β) if β is a shift of the word $(01)^{\mathbb{Z}}$. The relation R is regular and it is 2-valued so it has a regular 2-uniformisation. However, analogously as in Example 5, R has no shift-invariant uniformisation.

In an analogous way, Example 5 provides a relation witnessing that $1 \nLeftarrow 0$ in Theorem 8.

11 Conclusions

The main result of this paper gives effective characterisations of those regular relations $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ which admit:

1. a regular finitary uniformisation;
2. a shift-invariant uniformisation;
3. a regular uniformisation.

All the three questions are characterised by certain properties of a bi-Wilke algebra recognising R: the first one by the ability to choose witnesses for idempotents ($e \in E$); the second by the ability to choose witnesses for all elements ($k \in K$); and the third by this last requirement ($k \in K$) together with an additional condition allowing to choose witnesses inside groups (Γ for G). Surprisingly, although the condition about witnesses for all elements ($k \in K$) seems to be the most natural strengthening of the one for idempotents ($e \in E$), it exactly characterises the existence of (possibly non-regular) shift-invariant uniformisations.

Clearly, every regular uniformisation is also a shift-invariant uniformisation and at the same time a regular finitary uniformisation. However, clearly not every shift-invariant uniformisation is itself regular. Therefore, the implication between the condition from Proposition 24 and the one from Proposition 17 is quite unexpected. Another interesting consequence of our results is that every regular relation R that admits a regular finitary uniformisation, admits also a regular ℓ-uniformisation for some $\ell > 0$ (*i.e.* the cardinality of the sections is uniformly bounded).

This all provides a rather complete understanding of how to find regular uniformisations of relations between bi-infinite words. A general message of these results is that shift-invariance is not the only reasons why a given regular relation may not admit a regular uniformisation. These subtleties arise from the case of groups.

Our most important tool is Factorisation Lemma 7. It provides a way to understand the structure of a given bi-infinite word and to distinguish between words of three different types: simple words, where a single position can be easily defined; repetitive words, obtained by a $\mathbb{Z}$-indexed concatenation of factors of a fixed type k; and the most interesting case of words that can be split into a non-constant factorisation, with all the types of the factors coming from a group. We believe this lemma to be of independent interest.

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A Green's relations

Before we proceed to the remaining proofs of the paper, we need to recall the notions of Green's relations. Fix a semigroup S.

Definition 30. *Green's relations* over S are defined as follows:

$$x \leqslant_{\mathcal{J}} y \iff \exists s,t \in S.\ x = syt$$
$$x \leqslant_{\mathcal{R}} y \iff \exists s \in S.\ x = ys$$
$$x \leqslant_{\mathcal{L}} y \iff \exists s \in S.\ x = sy$$

Each of these relations induces an equivalence relation defined by $x =_{\mathcal{J}} y \iff x \leqslant_{\mathcal{J}} y \wedge y \leqslant_{\mathcal{J}} x$; analogously for $=_{\mathcal{R}}$ and $=_{\mathcal{L}}$. Finally, we say that $x =_{\mathcal{H}} y \iff x =_{\mathcal{L}} y \ \wedge\ x =_{\mathcal{R}} y$.

The equivalence classes of the above relations are called respectively $\mathcal{J}$-, $\mathcal{L}$-, $\mathcal{R}$-, and $\mathcal{H}$-classes of S. Directly from the definition we see that: each $\mathcal{R}$-class (resp. $\mathcal{L}$-class) is contained in a $\mathcal{J}$-class.

We will use the following standard facts about Green relations, see for instance [7, Annex A].

Fact 31. *Fix $s,t \in S$, such that $s =_{\mathcal{J}} t$. If $s \leqslant_{\mathcal{R}} t$ then $s =_{\mathcal{R}} t$. If $s \leqslant_{\mathcal{L}} t$ then $s =_{\mathcal{L}} t$*

Fact 32. *For every $s,t \in S$, if $s =_{\mathcal{H}} t =_{\mathcal{H}} st$ then $[s]_{\mathcal{H}}$ is a maximal (wrt. inclusion) subgroup of S.*

Fact 33. *Fix $s,t \in S$ such that $s =_{\mathcal{J}} t$. Then $st =_J s$ if and only if there is an idempotent in $[s]_{\mathcal{L}} \cap [t]_{\mathcal{R}}$.*

Claim 34. *For $s,t,r \in S$, if s, t, r, $s \cdot t$, and $t \cdot r$ belong to a common $\mathcal{J}$-class J, then $s \cdot t \cdot r \in J$ as well.*

Proof. By Fact 33, there exists an idempotent in $[t]_{\mathcal{L}} \cap [r]_{\mathcal{R}}$. By Fact 31, $s \cdot t =_{\mathcal{L}} t$, so $[s \cdot t]_{\mathcal{L}} = [t]_{\mathcal{L}}$. Hence $[s \cdot t]_{\mathcal{L}} \cap [r]_{\mathcal{R}}$ contains an idempotent. Now, the thesis follows directly from Fact 33. □

B Definability in MSO

In this short appendix we illustrate how to define the basic concepts used across the paper within the logic MSO.

First observe that we can simulate in MSO over bi-infinite words the quantification over words $\beta \in B^{\mathbb{Z}}$. Let $B = \{b_1, \ldots, b_n\}$. Whenever we write $\exists \beta \in B^{\mathbb{Z}}.\ \psi$ we can instead write

$$\exists X_{b_1}, \ldots, X_{b_n}.\ \mathrm{partition}(X_{b_1}, \ldots, X_{b_n}) \wedge \psi',$$

where $\mathrm{partition}(X_1, \ldots, X_n)$ says that these sets form a partition of the domain, and the formula ψ' is obtained from ψ by replacing each predicate $b_i(x)$ by the condition $x \in B_{b_i}$.

Now observe that the following properties are MSO-definable:

$$\text{"}V = [x,y)\text{"} \equiv \forall z.\ \big(z \in V \leftrightarrow x \leqslant z < y\big)$$
$$\mathrm{factor}(V) \equiv \exists x < y.\ V = [x,y)$$
$$\mathrm{factorisation}(X) \equiv \forall y.\ \exists x,z \in X.\ x < y < z$$
$$\begin{aligned}\mathrm{factor}(V,X) \equiv\ & \exists x,y \in X.\ x < y \wedge V = [x,y) \wedge \\ & \mathrm{factorisation}(X) \wedge \\ & \forall z \in X.\ z \in V \to z = x\end{aligned}$$

Additionally, MSO is able to evaluate types of factors of a word within a semigroup S. Consider a finite semigroup S and a homomorphism $h\colon A^+ \to S$. Then, for every $s \in S$ there exists a formula $\mathrm{value}_s(V)$ that holds form some factor $V = [x,y)$ over a word $\alpha \in A^{\mathbb{Z}}$ if and only if $h\big(\alpha{\upharpoonright}_V\big) = s$:

$$\begin{aligned}\mathrm{value}_s(V) \equiv\ & \exists x < y.\ V = [x,y) \wedge \\ & \exists \rho \in S^{\mathbb{Z}}.\ \rho(x) = h(\alpha(x)) \wedge \\ & \forall z > x.\ \rho(z) = \rho(z-1) \cdot h(\alpha(z)) \wedge \\ & \rho(y-1) = s\end{aligned}$$

Notice that the third line of the above formula requires us to hard-code the multiplication table of the semigroup S inside the formula.

The above formula can be easily used to define the values of the factors of a factorisation X as a labelling of X:

$$\begin{aligned}\mathrm{value}_s(x,X) \equiv\ & \mathrm{factorisation}(X) \wedge x \in X \wedge \\ & \exists y \in X.\ \mathrm{factor}([x,y),X) \wedge \\ & \mathrm{value}_s([x,y))\end{aligned}$$

We hope that the above basic formulae indicate how one can build the more complex formulae used across the paper.

C Proof of Proposition 10

This section is devoted to a proof of Proposition 10.

Proposition 10 ($*$). *Let $\varphi(x)$ be an MSO formula with one free variable x and let $\alpha \in A^{\mathbb{Z}}$ be a universal word. Assume that $\varphi(x)$ holds over α for some position $x \in \mathbb{Z}$. Then there exists a pattern (u,a,v) such that for every position $x' \in \mathbb{Z}$ if the pattern (u,a,v) appears in α at a position $x' \in \mathbb{Z}$ then $\varphi(x')$ holds over α.*

This fact follows from an analogous result for infinite words. An infinite word $w \in A^{\omega}$ is called *universal* if it contains infinitely many occurrences of every possible infix $u \in A^+$.

Lemma 35. *Let $L \subseteq A^{\omega}$ be a regular language of infinite words. If L contains some universal word, then there is some finite word $u \in A^+$ such that L contains all universal words starting with u.*

The following proof of Lemma 35 is a standard application of Wilke algebras and Green's relations.

Proof. Fix a Wilke algebra (W_{fin}, W_{inf}) that recognises L via a surjective homomorphism $h\colon (A^+, A^\omega) \to (W_{fin}, W_{inf})$.

Consider a universal infinite word $w \in L$ and let $w = u_0 u_1 \dots$ be its Ramsey decomposition w.r.t h, i.e. $h(u_0) = s$, $h(u_i) = e$ for $i > 0$, $s \cdot e = s$, and $e \cdot e = e$. Since w is universal, it contains every word as an infix infinitely many times. Therefore, e belongs to the $\leqslant_{\mathcal{J}}$-minimal $\mathcal{J}$ class of W_{fin}. As $s \cdot e = s$, also $s =_{\mathcal{J}} e$.

Fix $u = u_0$ and take any other universal word of the form uw' for some $w' \in A^\omega$. Our aim is to show that $uw' \in L$. Take a Ramsey decomposition of w' w.r.t h of the form $w' = v_0 v_1 \dots$ Let (s', e') be the linked pair (see [7, Section 2.2]) corresponding to this decomposition. Then, $s =_{\mathcal{J}} s' =_{\mathcal{J}} e =_{\mathcal{J}} e'$. Moreover, uw' admits a Ramsey decomposition for the linked pair (ss', e').

Since $ss' =_{\mathcal{J}} s$ and $ss' \leqslant_{\mathcal{R}} s$, we know that $ss' =_{\mathcal{J}} s$ (see Fact 31). Therefore, Proposition 2.17 from [7] implies that the linked pairs (s, e) and (ss', e') are conjugate in the sense of the definition from page 79 there. This implies by Corollary 2.9 there that $s \cdot e^\omega = ss' \cdot e'^\omega$ in W_{inf}. But $h(w) = s \cdot e^\omega$ and $h(uw') = ss' \cdot e'^\omega$ and since $w \in L$ also $uw' \in L$. ◻

We now show how Proposition 10 follows from Lemma 35.

We use the following lemma, which is also standard. It follows *e.g.* from the equivalence (3) ⇔ (4) in [7, Proposition 2.3 in Chapter IX].

Lemma 36. *For any* MSO *formula $\varphi(x)$ over bi-infinite words there is a number $k \geqslant 0$, regular languages $L^i \subseteq A^{-\omega}$, $N^i \subseteq A^\omega$, and letters $a_i \in A$, for $i = 1, \dots, k$, such that the following conditions are equivalent for all $w \in A^{-\omega}$, $w' \in A^\omega$, and $a \in A$:*

1. *$\phi(x)$ holds at the distinguished position labelled a in $w \cdot a \cdot w'$,*
2. *there is some $i \in \{1, \dots, k\}$ such that $a_i = a$, $w \in L^i$, and $w' \in N^i$.*

Proof of Proposition 10. Assume that $\varphi(x)$ holds in some universal word α at some position x. Without loss of generality, we can assume that $\alpha = w \cdot a \cdot w'$ and the distinguished position x is the position labelled a. This means that $w \in L^i$, $a = a_i$, and $w' \in N^i$ for some i.

As N^i contains a universal word, namely w', by Lemma 35 there is some finite word v such that N^i contains all right-infinite universal words starting with v. By the symmetry, there is some finite word u such that L^i contains all left-infinite universal words ending with u. Consider the pattern (u, a, v), we claim that it satisfies the requirements of Proposition 10.

Consider any $x \in \mathbb{Z}$ such that the pattern (u, a, v) appears in α at the position x. Let $\bar{w}a\bar{w}'$ be the corresponding decomposition of α' at the position x, *i.e.* α is a shift of $\bar{w}a\bar{w}'$. We know that u is a suffix of the left-infinite word $\bar{w}$ and v is a prefix of the right-infinite word $\bar{w}'$. Therefore, by the choice of the words u and v above, we know that $\bar{w} \in L^i$ and $\bar{w}' \in N^i$. This means that the formula φ holds over α at the position x. ◻

D Proof of Lemma 19

Lemma 19 (∗). *Let $R \subseteq A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ be a regular relation and let $\phi(x)$ be an* MSO *formula such that $\phi(x)$ defines a single position in each $\alpha \in \mathrm{dom}(R)$. Then the relation R admits a regular uniformisation.*

This lemma is a simple application of Theorem 3.

Given a pair of languages $L \subseteq A^{-\omega}$ and $N \subseteq A^\omega$ together with a letter $a \in A$, by $L \cdot a \cdot N$ we will denote the set of bi-infinite words that are shifts of a word of the form $w \cdot a \cdot w'$ where $w \in L$ and $w' \in N$.

Proof. Let $\psi(x)$ be an MSO formula over the alphabet $A \times B$ which holds in a word $\langle \alpha, \beta \rangle$ at a position $x \in \mathbb{Z}$ if and only if ϕ holds at a position x in α and $(\alpha, \beta) \in R$.

Then for every $(\alpha, \beta) \in R$ there is exactly one position x such that ψ holds at x in $\langle \alpha, \beta \rangle$, and for $(\alpha, \beta) \notin R$, no such position exists.

Let $a_1, \dots, a_k \in A \times B$ be letters and $L^1, \dots, L^k \subseteq (A \times B)^{-\omega}$ and $N^1, \dots, N^k \subseteq (A \times B)^\omega$ be languages as in Lemma 36 applied to $\psi(x)$. For $i = 1, \dots, k$, let

$$R^i_- = \{(w, w') \in A^{-\omega} \times B^{-\omega} \mid \langle w, w' \rangle \in L^i\}$$
$$R^i_+ = \{(w, w') \in A^{-\omega} \times B^{-\omega} \mid \langle w, w' \rangle \in N^i\}$$

be the relations corresponding to by L^i and N^i, respectively.

Each of the above relations admits a regular uniformisation, by Theorem 3. Let $F^i_- \subseteq R^i_-$ and $F^i_+ \subseteq R^i_+$ be the respective uniformisations. Clearly, for each $i = 1, \dots, k$ the relation corresponding to the language $F^i_- \cdot a_i \cdot F^i_+$ is a regular uniformisation of the relation corresponding to $R^i_- \cdot a_i \cdot R^i_+$.

We will now define $F \subseteq R$ as the relation containing a pair $(\alpha, \beta) \in A^{\mathbb{Z}} \times B^{\mathbb{Z}}$ if and only if the following conditions hold. Firstly, α must belong to the projection onto $A^{\mathbb{Z}}$ of one of the languages $F^i_- \cdot a_i \cdot F^i_+$. Let i be the minimal index for which it holds. Then $\langle \alpha, \beta \rangle$ must belong to $F^i_- \cdot a_i \cdot F^i_+$. It is easy to check that $F \subseteq R$. Moreover, if $\alpha \in \mathrm{dom}(R)$ then α belongs to the projection of at least one of the languages $R^i_- \cdot a_i \cdot R^i_+$. This means that $\mathrm{dom}(F) = \mathrm{dom}(R)$. Finally, by the assumptions on $\phi(x)$ we know that F is a functional relation: the single position defined by $\phi(x)$ is chosen uniquely based on the given word α and then the rest of the construction provides a unique word β such that $(\alpha, \beta) \in F$. ◻

E Proof of Factorisation Lemma 7

Lemma 7 ((∗) Factorisation Lemma)**.** *Let A be a finite alphabet and $h\colon A^+ \to S$ be a homomorphism to a finite semigroup S. There effectively exists an* MSO *formula $\varphi(x)$ with the following properties. Take any word $\alpha \in A^{\mathbb{Z}}$. Then, exactly one of the following conditions holds:*

1. *$\varphi[\alpha]$ is a singleton;*
2. *$\varphi[\alpha]$ is an s-constant factorisation of α for some $s \in S$;*
3. *$\varphi[\alpha]$ is a G-group factorisation of α for some group $G \subseteq S$.*

Moreover, if S is aperiodic (contains only singleton groups), then $\varphi(x)$ is a formula of first-order logic — $\varphi(x)$ does not use second-order quantifiers.

The rest of this section is devoted to a proof of that lemma. First, observe that we can substitute every letter $a \in A$ of a given word α by the respective value $h(a) \in S$ in the semigroup. Therefore, without loss of generality we can work with words $\alpha \in S^{\mathbb{Z}}$ and the homomorphism $[\![.]\!]$ that computes the product in S of a given word in S^+.

For the rest of this section we fix a finite semigroup S.

E.1 Aperiodic semigroups

A semigroup S is called *aperiodic* if whenever $G \subseteq S$ is a group then $|G| = 1$. This class of semigroups correspond to first-order logic (denoted FO), which is a fragment of MSO where only first-order quantifiers are allowed. In the same way as for MSO, we can say that a language or a relation is FO-definable.

The following theorem expresses the famous results of Schutzenberger [11] and McNaughton–Papert [6].

Theorem 37. *A language $L \subseteq A^+$ is* FO*-definable if and only if L is recognised by a homomorphism $h: A^+ \to S$ into a finite aperiodic semigroup.*

Analogous characterisations are known for other types of words, in particular for infinite words [12], etc.

E.2 Quotients

Take a bi-infinite word $\alpha \in S^{\mathbb{Z}}$ and let $X \subseteq \mathbb{Z}$ be a factorisation (*i.e.* a bi-unbounded set). Let

$$X = \{\dots, \iota_X(-1), \iota_X(0), \iota_X(1), \dots\}$$

be some fixed enumeration of X such that $\iota_X(x) < \iota_X(x+1)$ for every $x \in \mathbb{Z}$. Let $\alpha/X \in S^{\mathbb{Z}}$ be the bi-infinite word defined as $(\alpha/X)(x) \stackrel{\text{def}}{=} [\![\alpha\restriction_{[\iota_X(x), \iota_X(x+1))}]\!]$, for $x \in \mathbb{Z}$. We call α/X the *quotient* of α by X.

In other words, the positions x of the word α/X correspond to the factors V_x of the factorisation X, and the label of $(\alpha/X)(x)$ is the product $[\![\alpha\restriction_{V_x}]\!]$ of the corresponding factor of α.

Without loss of generality we can assume that the enumeration ι_X above is chosen in such a way that for $X = \mathbb{Z}$ we have $\iota_X(x) = x$ and therefore $\alpha/X = \alpha$. It means that $X = \mathbb{Z}$ is a *trivial* factorisation that corresponds to the identity quotient.

We will say that a factorisation Y is *coarser* than a factorisation X if $Y \subseteq X$. Notice that in that case, α/Y is (up-to shifts) a quotient of the word α/X.

E.3 Smooth words

A word $\alpha \in S^{\mathbb{Z}}$ is called *constant*, if α exactly one letter $s \in S$ appears in α, *i.e.* α is constant as a function.

We say that a word $\alpha \in S^{\mathbb{Z}}$ is $\mathcal{J}$*-smooth* (resp. $\mathcal{L}$-, $\mathcal{R}$-, or $\mathcal{H}$-smooth) if the values $\left[\!\left[\alpha\restriction_{[x,y)}\right]\!\right]$ for all $x < y$ belong to a common $\mathcal{J}$-class (resp. $\mathcal{L}$-, $\mathcal{R}$-, or $\mathcal{H}$-class).

The above requirement implies that in particular all the letters of the word α share a joint Green's class. But the condition is much stronger, because also each product of these letters must belong to that class.

The following fact means that once a smooth word is found, it will stay smooth, no matter what further factorisations we apply.

Remark 38. *Fix a $\mathcal{J}$-smooth (resp. $\mathcal{L}$-, $\mathcal{R}$-, or $\mathcal{H}$-smooth) word α and let $X \subseteq \mathbb{Z}$ be any factorisation. Then α/X is also $\mathcal{J}$-smooth (resp. $\mathcal{L}$-, $\mathcal{R}$-, or $\mathcal{H}$-smooth).*

Notice that the above remark does not work in the case of constant words: there may be a constant word α and a factorisation X such that α/X is not constant.

Clearly, the notions of constant and smooth words are invariant under shifts: if α is a shift of α' and one of them is, say, $\mathcal{J}$-smooth then the other one is $\mathcal{J}$-smooth as well.

E.4 Overview of the proof

Our approach to proving Lemma 7 is as follows. Given a word $\alpha_0 \in S^{\mathbb{Z}}$ we will define a sequence of coarser and coarser factorisations $X_1 \supseteq X_2 \supseteq \dots \supseteq X_n$ that can be sequentially applied to the word α_0 by putting $\alpha_{i+1} \stackrel{\text{def}}{=} \alpha_i/X_{i+1}$. The factorisations are chosen in a way to make the successive word α_{i+1} *more smooth*. The invariant of this construction is defined in terms of the set of $\mathcal{J}$-classes of letters that appear in α_i. Due to that invariant, we are guaranteed to reach a $\mathcal{J}$-smooth word α_n for a fixed n (in fact $n \leqslant |S|$).

Once a $\mathcal{J}$-smooth word is obtained, two additional factorisations $X_{n+1} \supseteq X_{n+2}$ allow us to obtain an $\mathcal{H}$-smooth word α_{n+2}. Due to Fact 32, the factorisation X_{n+2} is either a k-constant factorisation, or a G-group factorisation.

At each stage of the construction we must consider three cases: either the current word α_i is sufficiently *homogeneous* to define the consecutive factorisation; or it is constant and we don't need to do anything; or α_i is not constant and not *homogeneous*, in which case it allows us to define a single position of α. The former two cases halt the whole procedure, making Lemma 7 true due to either the first or the second possibility.

There are two additional difficulties of that construction. First, we need to define the construction in a uniform way, for all words α at once. Moreover, this uniform definition needs to be implemented as an MSO (or even FO) formula $\varphi(x)$. This leads to the definition of *transformations*.

E.5 Transformations

A *transformation* is a function $\Phi: A^{\mathbb{Z}} \to \mathrm{P}(\mathbb{Z})$ that assigns to each word $\alpha \in S^{\mathbb{Z}}$ a set of positions $\Phi(\alpha) \subseteq \mathbb{Z}$ such that $\Phi(\alpha)$ is either a single position or a factorisation.

We say that a transformation Φ is MSO-*definable* (resp. FO-*definable*) if there exists a formula $\varphi(x)$ of MSO (resp. FO) such that for every word $\alpha \in S^{\mathbb{Z}}$ we have

$$\Phi(\alpha) = \varphi[\alpha],$$

where $\varphi[\alpha] = \{x \in \mathbb{Z} \mid \varphi(x) \text{ holds over } \alpha \text{ at } x\}$.

We will now show how to compose transformations, using the quotients α/X and enumerations ι_X from Subsection E.2. Take two transformations Φ and Ψ. The composition $(\Psi \circ \Phi)$ is defined to be a function which on a word α is defined as follows:

1. If $\Phi(\alpha)$ is a single position x of the word α then return the same position x.
2. Otherwise, $X = \Phi(\alpha)$ is a factorisation of the word α. Let $\alpha' = \alpha/X$. If $X' = \Psi(\alpha')$ is a single position x' of α' then return $\iota_X(x')$, *i.e.* the corresponding position of the word α.
3. Otherwise, X' defined above is a factorisation of α'. Return the factorisation $\{\iota_X(x') \mid x' \in X'\}$ of α.

Notice that the above definition guarantees that $(\Psi\circ\Phi)(\alpha)$ is a subset of $\Phi(\alpha)$. This subset (if bi-unbounded) is a coarser factorisation than $\Phi(\alpha)$. The above definition is compositional with the quotients α/X in the following technical sense.

Fact 39. *Take two transformations Φ and Ψ and assume that for some word $\alpha \in S^{\mathbb{Z}}$ the result $X'' = (\Psi \circ \Phi)(\alpha)$ is a factorisation. Let $X = \Phi(\alpha)$, $\alpha' = \alpha/X$, and $X' = \Psi(\alpha')$. Then the following two words are shift-equivalent:*

$$\alpha'/X' \sim \alpha/X''. \tag{5}$$

This means that up-to shifts and the case when the result is a single position, composition of transformations behaves like composition of functions.

E.6 MSO-definability

Our aim is to prove the following lemma.

Lemma 40. *If Φ and Ψ are two* MSO*-definable transformations then the transformation $(\Psi \circ \Phi)$ is also* MSO*-definable.*

Proof. The proof of this lemma is analogous to the argument in Fact 27. Assume that $\phi(x)$ and $\psi(x)$ are two formulae witnessing that Φ and Ψ are MSO-definable, respectively. We aim at constructing a formula $\theta(x)$ such that $\theta[\alpha] = (\Psi \circ \Phi)(\alpha)$ for every word $\alpha \in S^{\mathbb{Z}}$.

First, let $\theta(x)$ check if $\psi(x)$ is satisfied for a single position x_0. If it is the case, $\theta(x)$ holds exactly for that one position x_0. Otherwise, let X be the set of positions where $\psi(x)$ holds — whenever we check whether $x \in X$, instead we can check whether $\psi(x)$ holds.

Now, for every position $x \in X$ we need to define in MSO the value $[\![\alpha{\restriction_{V_x}}]\!]$ of the unique factor V_x of the factorisation X that contains x. This follows from the fact that for each $s \in S$ the language L_s of finite words u such that $[\![u]\!] = s$ is MSO-definable. Therefore, to check if $[\![\alpha{\restriction_{V_x}}]\!]$ it is enough to check if $\alpha{\restriction_{V_x}} \in L_s$.

Finally, $\theta(x)$ can evaluate the formula $\phi(x)$ relativised to the set X and the labelling given by the evaluations $[\![\alpha{\restriction_{V_x}}]\!]$. This way, if X is a factorisation and $x \in X$ then $\theta(x)$ holds if and only if $\iota_X^{-1}(x) \in \Psi(\alpha/X)$. □

Corollary 41. *If Φ and Ψ are two* FO*-definable transformations and the semigroup S is aperiodic then the transformation $(\Psi \circ \Phi)$ is also* FO*-definable.*

Proof. The proof of this corollary is exactly the same as the proof of Lemma 40. The only difference is that when one wants to define the label of the position $x \in X$ according to the formula $[\![\alpha{\restriction_{V_x}}]\!]$, one needs to rely on the fact (cf. Theorem 37) that the language L_s used above is FO-definable because the semigroup S is aperiodic. This is the reason for the additional assumption about S in the statement of the corollary. □

E.7 Basic predicates

We will now define a family of basic FO formulae that allow to distinguish certain positions of a given word. For each non-empty set $K \subseteq S$, choose an element $k \in K$, called the *witness* for K.

Let $\lambda(x)$ be the predicate which, over a word α, first computes the set $K \subseteq S$ of all letters that appear in α and then holds at x if and only if $\alpha(x) \in S$ is the chosen witness k for K.

Let $\eta(x)$ be the predicate which, over a word α, holds at a position x if and only if

$$\alpha(x) \cdot \alpha(x{+}1) <_{\mathcal{J}} \alpha(x) \;\vee\; \alpha(x) \cdot \alpha(x{+}1) <_{\mathcal{J}} \alpha(x{+}1),$$

where the products above are computed in the semigroup S.

Notice that both above formulae are expressible in FO.

Lemma 42. *For $\alpha \in S^{\mathbb{Z}}$, if $\eta[\alpha] = \varnothing$ then α is $\mathcal{J}$-smooth.*

Proof. Assume that $\eta[\alpha] = \varnothing$, *i.e.* $\eta(x)$ holds over α at no position $x \in \mathbb{Z}$. It means that there exists a fixed $\mathcal{J}$-class J of S such that for every $x \in \mathbb{Z}$ we have

$$\alpha(x) =_{\mathcal{J}} \alpha(x) \cdot \alpha(x{+}1) =_{\mathcal{J}} \alpha(x{+}1) \in J, \tag{6}$$

where the middle value is the product of the two letters in the semigroup S.

Assume for the sake of contradiction that there is some factor $[x, y)$ with $x < y$ such that $[\![\alpha{\restriction_{[x,y)}}]\!] \notin J$. Assume additionally that $[x, y)$ is a minimal such factor with respect to inclusion.

Equation (6) implies that $y \geqslant x + 3$. It means that $[x, y - 2)$ is a non-trivial factor. Let $s = [\![\alpha{\restriction_{[x,y-2)}}]\!]$, $t = \alpha(y-2)$ and $r = \alpha(y-1)$. Then $[\![\alpha{\restriction_{[x,y)}}]\!] = s \cdot t \cdot r \notin J$. However, all the values s, t, r, $s \cdot t$, and $t \cdot r$ can be obtained as the values of factors $V' \subsetneq [x, y)$ in α. Therefore, by the minimality of $[x, y)$ all these values belong to J. This gives us a contradiction, because Claim 34 implies that $s \cdot t \cdot r \in J$. □

E.8 Lifting predicates

The two formulae $\lambda(x)$ and $\eta(x)$ defined above may not give rise to a transformation, because the sets $\lambda[\alpha]$ and $\eta[\alpha]$ may not be singleton nor bi-infinite. To overcome this obstacle, we will now show how to lift an arbitrary FO formula $\phi(x)$ into an FO-definable transformation $\Gamma(\phi)$ that *groups* pieces of the word that satisfy $\phi(x)$ with those that do not satisfy it.

More formally, given an input word $\alpha \in S^{\mathbb{Z}}$, define $\Gamma(\phi)(\alpha)$ as follows.

(A) If there exist any of the following:
- the left-most position in $\phi[\alpha]$
- the right-most position in $\phi[\alpha]$
- the left-most position in $(\neg\phi)[\alpha]$
- the right-most position in $(\neg\phi)[\alpha]$

then return the left-most position satisfying any of the conditions above.

(B) If $\phi[\alpha] = \mathbb{Z}$ or $\phi[\alpha] = \varnothing$ then return the trivial factorisation into one-letter factors $X = \mathbb{Z}$.

(C) Otherwise, return the factorisation X such that $x \in X$ if and only if $\phi(x)$ holds but $\phi(x-1)$ does not hold.

The most important third case above is designed in such a way to guarantee that each factor V of the returned factorisation contains both: a position x where $\phi(x)$ holds, and a position x' where $\neg\phi(x')$ holds. This implies the following remark.

Remark 43. *If Case (C) holds in $\Gamma(\phi)(\alpha)$ then each factor V of the constructed factorisation X satisfies $|V| \geqslant 2$.*

Notice that if ϕ is an FO formula then $\Gamma(\phi)$ is an FO-definable transformation. In particular $\Gamma(\lambda)$ and $\Gamma(\eta)$ are both FO-definable.

E.9 Fixing the $\mathcal{J}$-class

We will now define a transformation Δ that will perform the inductive step of our construction, ultimately giving us a $\mathcal{J}$-smooth word.

Given a word $\alpha \in S^{\mathbb{Z}}$, define $\Delta(\alpha)$ as follows.

(I) If α is constant then return $X = \mathbb{Z}$.
(II) If $\eta[\alpha] = \varnothing$ then return $X = \mathbb{Z}$.
(III) If $\eta[\alpha] = \mathbb{Z}$ then return $\Gamma(\lambda)(\alpha)$.
(IV) Otherwise return $\Gamma(\eta)(\alpha)$.

It is easy to see that Δ is an FO-definable transformation.

Given a word $\alpha \in S^{\mathbb{Z}}$ by $J_{\downarrow}(\alpha) \subseteq S$ we will denote the set of elements $t \in S$ such that for some position $x \in \mathbb{Z}$ we have $t \leqslant_{\mathcal{J}} \alpha(x)$, *i.e.* the $\mathcal{J}$-downward-closure of the set of letters that appear in α.

The invariant of our construction will be that the set $J_{\downarrow}(\alpha)$ decreases in the inclusion order, as expressed by the following lemma.

Lemma 44. *Fix a non-constant word $\alpha \in S^{\mathbb{Z}}$. Assume that $X = \Delta(\alpha)$ is a factorisation (i.e. not a single position) and put $\alpha' = \alpha/X$. Then $J_{\downarrow}(\alpha') \subseteq J_{\downarrow}(\alpha)$. Moreover, if $J_{\downarrow}(\alpha') = J_{\downarrow}(\alpha)$ then α is $\mathcal{J}$-smooth.*

Proof. The inclusion $J_{\downarrow}(\alpha') \subseteq J_{\downarrow}(\alpha)$ follows from the fact that for any factorisation X of a word α we have $J_{\downarrow}(\alpha/X) \subseteq J_{\downarrow}(\alpha)$, because $s \cdot t \leqslant_{\mathcal{J}} s$ and $s \cdot t \leqslant_{\mathcal{J}} t$ for every $s, t \in S$.

For the second part, assume that α is a non-constant word. Notice that if $\eta[\alpha] = \varnothing$ then by Lemma 42 we know that α is $\mathcal{J}$-smooth. Therefore, consider the case that $\eta[\alpha] \neq \varnothing$. Let $X = \Delta(\alpha)$ be the factorisation returned by $\Delta(\alpha)$. Take a $\mathcal{J}$-maximal $\mathcal{J}$-class $J \subseteq S$ in $J_{\downarrow}(\alpha)$. We want to show that no element $s \in J$ appear in α'. This implies that $J_{\downarrow}(\alpha') \neq J_{\downarrow}(\alpha)$ and concludes the proof of Lemma 44.

Fix a $\mathcal{J}$-class J as above and take a factor $[x, y)$ of the factorisation X. We need to show that $[\![\alpha\restriction_{[x,y)}]\!] \notin J$.

We will first argue that $\eta(x)$ holds over α. Firstly, if $\eta[\alpha] = \mathbb{Z}$ then it is the case. Otherwise, $X = \Gamma(\eta)(\alpha)$ and by the definition of $\Gamma(\eta)$ we know that each factor begins with a position satisfying $\eta(x)$.

We will now argue that $|V| \geqslant 2$. In both the above cases it follows from Remark 43. Firstly, if $\eta[\alpha] = \mathbb{Z}$ then we use the fact that α is not a constant word and therefore $\lambda[\alpha]$ is neither $\varnothing$ nor $\mathbb{Z}$. If $\eta[\alpha] \neq \mathbb{Z}$ then we invoke Remark 43 for $\Gamma(\eta)$.

Now, since $\eta(x)$ holds we know that $\alpha(x) \cdot \alpha(x{+}1) <_{\mathcal{J}} \alpha(x')$ for x' equal either x or $x{+}1$. Since $|V| \geqslant 2$, we know that $x{+}1 < y$ and therefore $[\![\alpha\restriction_{[x,y)}]\!] \leqslant_{\mathcal{J}} \alpha(x) \cdot \alpha(x{+}1) <_{\mathcal{J}} \alpha(x') \in J_{\downarrow}(\alpha)$. Therefore, by the maximality of J in $J_{\downarrow}(\alpha)$, we know that $[\![\alpha\restriction_{[x,y)}]\!] \notin J$. □

Given a transformation Φ and a number $n \geqslant 1$, by Φ^n we will denote the nth composition of Φ, defined inductively as $\Phi^1 = \Phi$ and $\Phi^{n+1} = \Phi \circ \Phi^n$. For $n = 0$ the transformation Φ^0 is the trivial transformation $\Phi^0(\alpha) = \mathbb{Z}$.

Corollary 45. *Take a word $\alpha \in S^{\mathbb{Z}}$ and let $n = |S|$. If $\Delta^n(\alpha)$ is a factorisation X then α/X is either constant or $\mathcal{J}$-smooth.*

Proof. First notice that the definition of compositions of transformations implies that if any of the sets $\Delta^i(\alpha)$ for $i = 1, \ldots, n$ is a single position then $\Delta^n(\alpha)$ is also a single position, which contradicts the assumption.

Put $X_i = \Delta^i(\alpha)$ and $\alpha_i = \alpha/X_i$ for $i = 1, \ldots, n$ and let $\alpha_0 = \alpha$. Equation (5) in Fact 39, when inlined, takes the form

$$(\alpha/\Phi(\alpha))/\Psi(\alpha/\Phi(\alpha)) \;\sim\; \alpha/(\Psi \circ \Phi)(\alpha).$$

Take $i = 1, \ldots, n$ and apply the above equivalence to $\Phi = \Delta^{i-1}$ and $\Psi = \Delta$:

$$(\alpha/\Delta^{i-1}(\alpha))/\Delta(\alpha/\Delta^{i-1}(\alpha)) \sim \alpha/\Delta^i(\alpha),$$

which means that

$$\alpha_{i-1}/\Delta(\alpha_{i-1}) \;\sim\; \alpha_i.$$

The definition of Δ guarantees that if for some $i = 0, \ldots, n$ the word α_i is constant then α_n is also constant. Remark 38 implies that if any of the words α_i for $i = 0, \ldots, n$ is $\mathcal{J}$-smooth then so is α_n. Therefore, for the rest of the proof we can assume that each of the words α_i for $i = 0, \ldots, n$ is defined, non-constant, and not $\mathcal{J}$-smooth.

Under these assumptions, Lemma 44 implies that $J_{\downarrow}(\alpha_i)$ for $i = 0, \ldots, n$ is a strictly-decreasing sequence of non-empty subsets of S. This is a contradiction, because $n = |S|$. ◻

E.10 Fixing the $\mathcal{L}$- and $\mathcal{R}$-class

Lemma 46. *If $\alpha \in S^{\mathbb{Z}}$ is a non-constant, $\mathcal{J}$-smooth word and $\Gamma(\lambda)(\alpha)$ returns a factorisation X then $\alpha' = \alpha/X$ is $\mathcal{R}$-smooth.*

Proof. Let K be the set of letters that appears in α and let $k \in K$ be the witness for K as in the definition of $\lambda(x)$, see Subsection E.7. Consider a factor $[x', y')$ with $x' < y'$ of the word α' and let $s = [\![\alpha\restriction_{[x',y')}]\!]$. We will show that $s =_{\mathcal{R}} k$.

Let $x, y \in X$ be the positions of the word α that correspond to the positions x' and y' of α', *i.e.* $x = \iota_X(x')$ and $y = \iota_X(y')$. The definition of α/X together with associativity of $\cdot$ imply that $[\![\alpha\restriction_{[x,y)}]\!] = s$.

Looking at the definition of $\Gamma(\lambda)$ we see that under our assumptions that X is a factorisation and α is non-constant, $\Gamma(\lambda)(\alpha)$ must be defined by Case (C). Therefore, the fact that $x \in X$ implies that $\alpha(x) = k$. This means that $s \leqslant_{\mathcal{R}} k$. Since α is $\mathcal{J}$-smooth, $s =_{\mathcal{J}} k$. Thus, Fact 31 implies that $s =_{\mathcal{R}} k$. ◻

Our aim now is to repeat the above construction in such a way to fix the $\mathcal{L}$-class of the quotient. This requires us to concatenate factors *to the left*. This can be easily achieved using the definition of Γ: the transformation $\Gamma(\neg\phi)$ returns similar factorisations as $\Gamma(\phi)$, but aligned to the right instead of left in the sense that if $X = \Gamma(\neg\phi)(\alpha)$ is a factorisation then the last position x of each of the factors V of X satisfies $\phi(x)$. By repeating the argument above, we get the following corollary.

Corollary 47. *If $\alpha \in S^{\mathbb{Z}}$ is a non-constant, $\mathcal{J}$-smooth word and $\Gamma(\neg\lambda)(\alpha)$ returns a factorisation X then $\alpha' = \alpha/X$ is $\mathcal{L}$-smooth.*

Notice that if $\alpha \in S^{\mathbb{Z}}$ is constant then both $\Gamma(\lambda)(\alpha)$ and $\Gamma(\neg\lambda)(\alpha)$ are trivial factorisations $X = \mathbb{Z}$.

E.11 Fixing the $\mathcal{H}$-class

We are now in position to define a transformation realising the goals of the formula $\varphi(x)$ from Factorisation Lemma 7. Define $\Phi \stackrel{\text{def}}{=} \Gamma(\neg\lambda) \circ \Gamma(\lambda) \circ \Delta^{|S|}$. By combining the two results of Subsection E.10 with Corollary 45 and Remark 38, we get the following observation.

Claim 48. *Take any word $\alpha \in S^{\mathbb{Z}}$. Let $X = \Phi(\alpha)$ be a factorisation. Then α/X is either constant or $\mathcal{H}$-smooth.*

This immediately implies Factorisation Lemma 7. Let $\varphi(x)$ be a formula that is obtained by the composition of the formulae realising all the transformations in Φ. Then, for every word $\alpha \in S^{\mathbb{Z}}$, the formula $\varphi(x)$ either defines a single position, or defines a factorisation X such that the word α/X is either constant, or $\mathcal{H}$-smooth.

If α/X is constantly equal $k \in S$ then the factorisation X is a k-constant factorisation. Fact 32 implies that if α/X is non-constant and $\mathcal{H}$-smooth then all the letters of α/X come from a group $G \subseteq S$ and therefore X is a G-group factorisation of α.

All the three involved transformations $\Gamma(\lambda)$, $\Gamma(\eta)$, and Δ are in fact FO-definable. Corollary 41 implies that, under the assumption that S is aperiodic, their composition is also FO-definable. If the semigroup S is not aperiodic, then the composition Φ is MSO-definable. This concludes the proof of Lemma 7.