Fano-Mori contractions of high length

Luca Tasin (joint work with M. Andreatta)
luca.tasin@unitn.it

UNIVERSITY OF TRENTO - Italy

Notation

Let X be an n-dimensional projective variety over $\mathbb{C}$ with mild singularities (all over the poster we will assume that X is terminal and $\mathbb{Q}$-factorial).

Let $N_1(X)$ be the real vector space of 1-cycles modulo the equivalence relation induced by the intersection of cycles.
We denote by $NE(X)$ the convex cone in $N_1(X)$ generated by effective 1-cycles and by $\overline{NE}(X)$ its closure.

A **Fano-Mori contraction** associated to an extremal ray $R \subset \overline{NE}(X)$ is a morphism $\varphi_R : X \to Z$ such that φ has connected fibres, Z is a normal variety and any curve $C \subset X$ is contracted by φ_R if and only if $[C] \in R$ and $K_X.C < 0$.

If $\dim X = \dim Z$ then φ is birational, otherwise is said to be of fibre type.
The set

$$E = \{x \in X : \varphi \text{ is not an isomorphism at } x\}$$

is called the exceptional locus of φ.

MMP

The goal of the Minimal Model Program is to associate to a variety X a minimal model (or a Mori fibre space) Y, which preserves most of the birational properties of X, but it is simpler to study.
This process is achieved by a sequence of Fano-Mori contractions (or flips associated to them).
That is why these maps are crucial in the birational classification of projective varieties.

The problem

Let $\varphi : X \to Z$ be a Fano-Mori contraction. Then there are an ample line bundle L and a positive rational number τ such that φ is supported by $K_X + \tau L$, that is φ is given by the linear system $|M(K_X + \tau L)|$ for $M \gg 0$.
Our aim is to study φ according to r.

Adjunction Theory

Classical Adjunction Theory studies polarized varieties, that is pair (X, L), where L is an ample line bundle on X. The origin of this theory goes back to the past century, when Castelnuovo and Enriques were studying projective surfaces by relating the geometry of a surface S to the geometry of its hyperplane sections.

The basic idea is to apply induction on a suitable element of $X' \in |L|$. By adjunction

$$(K_X + \tau L)_{|X'} = K_{X'} + (\tau - 1)L_{|X'}$$

and we would like to have a Fano-Mori contraction in one dimension less and then lift properties of (X', L') to (X, L).

References

[1] M. Andreatta and J. A. Wiśniewski. A note on nonvanishing and applications. *Duke Math. J.*, volume 72 n. 3, 1993, 739–755.

[2] M. Andreatta and L. Tasin. Fano-Mori contractions of high length on projective varieties with terminal singularities. Preprint, arXiv:1212.5075.

Our tools

Under appropriate conditions, the strategy of Adjunction Theory is feasible thanks to the following key results.

Theorem 1 ([1]). *Let $\varphi : X \to Z$ be a contraction supported by $K_X + \tau L$, such that Z is affine. If φ is birational and $\dim F \leq \tau + 1$ for any fibre F, then L is φ-base point free.*

Lemma 2 (Horizontal slicing). *Let $\varphi : X \to Z$ be a birational contraction supported by $K_X + \tau L$, such that Z is affine and $\tau \geq 1$. Let $X' \in |L|$ be a general divisor and let $\varphi' = \varphi_{|X'} : X' \to Z'$. If X' is normal, then φ' is a contraction supported by $K_{X'} + (\tau - 1)L_{|X'}$.*

Weighted blow-ups

Let $\sigma = (a_1, \ldots, a_k, 0, \ldots, 0) \in \mathbb{N}^n$ such that $a_i > 0$ and $\gcd(a_1, \ldots, a_k) = 1$. Let $M = \operatorname{lcm}(a_1, \ldots, a_k)$.
We denote by $\mathbb{P}(a_1, \ldots, a_k)$ the weighted projective space with weight $(a_1, \ldots, a_k)$.

Definition 3. *Let $X = \mathbb{A}^n = \operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]$ and $Z = \{x_1 = \ldots = x_k = 0\} \subset X$. Consider the rational map*

$$\varphi : \mathbb{A}^n \to \mathbb{P}(a_1, \ldots, a_k)$$

*given by $(x_1, \ldots, x_n) \mapsto (x_1^{a_1} : \ldots : x_k^{a_k})$.
The weighted blow-up of X along Z with weight σ is defined as the closure $\overline{X}$ in $\mathbb{A}^n \times \mathbb{P}(a_1, \ldots, a_k)$ of the graph of φ, together with the morphism $\pi : \overline{X} \to X$ given by the projection on the first factor.*

The map π is birational and contracts an exceptional irreducible divisor E to Z. Moreover for any point $z \in Z$ we have $\pi^{-1}(z) = \mathbb{P}(a_1, \ldots, a_k)$.

Definition 4. *Let X be a smooth variety and Z a smooth subvariety of codimension k. Let $\mathcal{I}_{\sigma,d}$ be ideal sheaves on X such that there is a covering $\{U_i \cong \mathbb{C}^n\}_{i \in I}$ on X so that for any $i \in I$ there are local coordinates $x_1, \ldots, x_n$ on U_i for which $Z \cap U_i = \{x_1 = \ldots = x_k = 0\}$ and*

$$\Gamma(U, \mathcal{I}_{\sigma,d}) = (x_1^{s_1} \cdots x_n^{s_n} : \sum_{j=1}^{k} s_j a_j \geq d).$$

A weighted blow-up of X along Z with weight σ is the projectivization

$$\pi : \overline{X} = \operatorname{Proj} \bigoplus_{d \geq 0} \mathcal{I}_{\sigma,d} \to X.$$

We call $\mathcal{I}_{\sigma,d}$ a σ-weighted ideal sheaf of degree d for Z in X.

The main results (see [2])

Theorem 5. *Let X be an n-dimensional projective variety with terminal singularities and let L be an ample Cartier divisor on X. Let $\varphi : X \to Z$ be a Fano-Mori contraction supported by $K_X + \tau L$. Assume that $\tau > n - 2$ and that f is birational. Then f is a weighted blow-up of a smooth point with weight $\sigma = (1, 1, b, \ldots, b)$, where b is a positive integer.*

Idea of proof for $n = 3$.
Let $X' \in |L|$ be general. By Theorem 1, X' is a smooth surface and $f' = f_{|X'} : X' \to Z'$ is a F-M contraction supported by $K_{X'} + (\tau - 1)L_{|X'}$. By Castelnuovo Theorem f' is a smooth blow-up and we have $f^* f_*(L) = L + bE$ for a positive integer b.
Applying Kawamata-Viehweg Vanishing we get the exact sequence

$$0 \to f_*\mathcal{O}_X(-(d-1)bE) \xrightarrow{\cdot x_n} f_*\mathcal{O}_X(-dbE) \to f_*\mathcal{O}_{X'}(-dbE) \to 0 \qquad (0.0.1)$$

from which we can then prove that

$$f_*\mathcal{O}_X(-dbE) = I_{\sigma,d} = (x_1^{s_1} x_2^{s_3} x_n^{s_3} : s_1 + s_2 + bs_3 \geq db).$$

Since $X = \operatorname{Proj} \bigoplus_{d \geq 0} f_*\mathcal{O}_X(-dbE)$, we are done.

$\square$

Theorem 6. *Let X be a normal projective variety with terminal singularities and let L be an ample Cartier divisor on X. Let $f : X \to Z$ be a Fano-Mori contraction supported by $K_X + \tau L$. Assume that $\dim E = n-1$ and that all fibres have dimension less or equal to $r + 1$. Set $C := f(E) \subset Z$.*

1. *Then $\operatorname{codim}_{Z'} C = r + 2$, there is a closed subset $S \subset Z$ of codimension al least 3 such that $Z' = Z \backslash S$ and $C' = C \backslash S$ are smooth, and $f' : X' = X \backslash f^{-1}(S) \to Z'$ is a weighted blow-up along C' with weight $\sigma = (1, 1, b \ldots, b, 0, \ldots, 0)$, where the number of b's is r.*
2. *Let $\mathcal{I}'$ be a σ-weighted ideal sheaf of degree b for $Z' \subset X'$ and let $i : Z' \to Z$ be the inclusion; let also $\mathcal{I} := i_*(\mathcal{I}')$ and $\mathcal{I}^{(m)}$ be the m-th symbolic power of $\mathcal{I}$. Then $X = \operatorname{Proj} \bigoplus_{m \geq 0} \mathcal{I}^{(m)}$.*