Zeta function of a dg-category

Evgeny Shinder
Institut für Mathematik
Johannes Gutenberg-Universität Mainz

joint work with Sergey Galkin

Grothendieck ring of varieties and the motivic zeta function

Let F be a field. $K_0(Var/F)$ is generated by symbols $[X]$ where X/F is an algebraic variety with relations

$$[X] = [X \backslash Z] + [Z]$$

for $Z \subset X$. $K_0(Var/F)$ has a ring structure with a product given by $[X] \cdot [Y] = [X \times Y]$.
Let $Sym^n(X)$ be a (usually singular) variety X^n/Σ_n. These operations define a λ-structure on $K_0(Var/F)$:

$$Sym^n(X) = \sum_{j=0}^{n} Sym^j(X \backslash Z) \cdot Sym^{n-j}(Z). \qquad (1)$$

The motivic zeta-function is defined by Kapranov as:

$$Z_{mot}(X,t) = \sum_{n=0}^{\infty} [Sym^n(X)] t^n \in K_0(Var/F)[[t]].$$

(1) is equivalent to

$$Z_{mot}(X,t) = Z_{mot}(Z,t) \cdot Z_{mot}(X \backslash Z, t).$$

$Z_{mot}(X,t)$ captures invariants such as the number of points over finite fields and the Euler characteristic.

Grothendieck ring of dg-categories and the categorical zeta function

Assume $char(F) = 0$. $K_0(dg - cat/F)$ is generated by symbols $[\mathcal{C}]$ where $\mathcal{C}$ is a smooth proper pretriangulated dg-category of finite type over F and relations

$$[\mathcal{C}] = [\mathcal{A}] + [\mathcal{B}]$$

for semiorthogonal decompositions $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$.
$K_0(dg - cat/F)$ is a ring with product (roughly speaking) given by $[\mathcal{C}] \cdot [\mathcal{D}] = [\mathcal{C} \otimes \mathcal{D}]$. $Sym^n(\mathcal{C})$ is defined as $\left(\mathcal{C}^{\otimes n}\right)^{\Sigma_n}$ (Kapranov-Ganter).
For example, if $\mathcal{C} = D^b(X)$, then $Sym^n(\mathcal{C})$ is the derived category of the n-th symmetric stack power of X.
One proves that Sym^n define a λ-structure on $K_0(dg - cat/F)$ and thus the categorical zeta-function

$$Z_{cat}(\mathcal{C},t) = \sum_{n=0}^{\infty} [Sym^n(\mathcal{C})] t^n \in K_0(dg - cat/F)[[t]].$$

is multiplicative for semi-orthogonal decompositions. If X is a smooth projective variety, we write $Z_{cat}(X,t) = Z_{cat}(D^b(X),t)$.

The relation between zeta-functions

According to Bondal-Larsen-Lunts and Bittner-Looijenga there exists a well-defined ring homomorphism

$$\mu_{dg} : K_0(Var/F) \to K_0(dg - cat/F)$$

which maps a smooth projective variety $[X]$ to its derived category $D^b(X)$. We expect the following formula to hold:

$$Z_{cat}(X,t) =^? \prod_{k \geq 1} \mu_{dg}(Z_{mot}(X,t^k)) \in K_0(dg - cat/F). \qquad (2)$$

Example: Varieties with full exceptional collections

If X admits a full exceptional collection of r objects, then $[D^b(X)] = r \in K_0(dg - cat/F)$, so that

$$Z_{cat}(X,t) = Z_{cat}(pt,t)^r,$$

thus we may assume $X = pt := Spec(F)$. In this case

$$Z_{cat}(pt,t) = \sum_{n \geq 0} [D^b(pt/\Sigma_n)] t^n$$

$D^b(pt/\Sigma_n)$ is the derived category of representations of Σ_n. Let p_k be the number of irreducible representations. We have

$$Z_{cat}(pt,t) = \sum_{n=0}^{\infty} p_n t^n = \prod_{k \geq 1} \frac{1}{1-t^k}$$

is the so-called partition function. We see that (2) indeed holds in this case since $Z_{mot}(pt,t) = \frac{1}{1-t} = 1 + t + t^2 + \ldots$.

Example: Götsche's formula

Let X/F be a smooth projective surface. By the McKay correspondence of Bridgeland-King-Reid and Hainman we have $Sym^n(D^b(X)) \simeq D^b(Hilb_n(X))$, so that

$$Z_{cat}(X,t) = \sum_{n=0}^{\infty} [D^b(Hilb_n(X))]\, t^n.$$

We use a formula of Götsche:

$$\sum_{n=0}^{\infty} [Hilb_n(X)] t^n = \prod_{k \geq 1} Z_{mot}(X, \mathbb{A}^{k-1} t^k),$$

and apply μ_{dg}:

$$Z_{cat}(X,t) = \sum_{n=0}^{\infty} [D^b(Hilb_n(X))] t^n = \prod_{k \geq 1} \mu_{dg}(Z_{mot}(X,t^k)).$$

and see that (2) holds here.

Possible applications

- Obstructions to existence of full exceptional collections
- Relation to motives, in particular Kimura finite-dimensionality