On Fourier-Mukai Type Functors

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A theorem of Orlov

Theorem [1, Orlov] Let X and Y be smooth projective varieties over an algebraically closed field k. Consider an exact functor
$$F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$$
If F is fully faithful and has a right adjoint, then there exists an object $E \in D^b_{Coh}(X \times Y)$ such that F is isomorphic to the Fourier-Mukai transform with kernel E.

The **Fourier-Mukai transform associated to a kernel** E is defined as
$$\Phi_E(-) = Rp_{2*}(E \overset{L}{\otimes} Lp_1^*(-))$$
An **exact functor** between two triangulated categories is a functor that is additive, commutes with shifts, and sends triangles to triangles.

...but more is true

All geometric functors are known to be isomorphic to a FM transform. For example:

- For $- \overset{L}{\otimes} \mathscr{C} : D^b_{Coh}(X) \to D^b_{Coh}(X)$, we can take the kernel $E = R\Delta_* \mathscr{C} \in D^b(X \times X)$
- For any $f : X \to Y$, the functor $Rf_* : D^b_{Coh}(X) \to D^b_{Coh}(Y)$ is isomorphic to the FM transform with kernel $\mathscr{O}_{\Gamma_f}$

Question:

What if F is not fully faithful?
No counterexamples are known!

A few developments

- **Theorem**[2, Bondal-Van den Bergh] Let X be a smooth projective variety over k. Every contravariant cohomological functor of finite type
$$H : D^b_{Coh}(X) \to \mathrm{Vect}_k$$
is representable by an object in $D^b_{Coh}(X)$.
This implies that **any** exact functor $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$ has a left and right adjoint.
- **Theorem** [3, Canonaco-Stellari] Let X and Y be smooth projective varieties over a field. Consider an exact functor $F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$ such that for any two sheaves $\mathscr{F}$ and $\mathscr{G} \in \mathrm{Coh}(X)$
$$\mathrm{Hom}_{D^b_{Coh}(Y)}(F(\mathscr{F}), F(\mathscr{G})[j]) = 0 \text{ if } j < 0$$
Then there exists an object $E \in D^b_{Coh}(X \times Y)$ such that F is isomorphic to the Fourier-Mukai transform with kernel E.

We can compute the cohomology sheaves of the prospective kernel

Even when we don't know if our functor F is isomorphic to a Fourier-Mukai functor with kernel E, we have the following:

Theorem Let X and Y be smooth projective varieties. Consider an exact functor
$$F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$$
There exists a sequence of sheaves $\mathscr{B}^M, \ldots, \mathscr{B}^N$ on $X \times Y$ and maps
$$\mathscr{H}^i(F(\mathscr{E}(n))) \to p_{2*}(\mathscr{B}^i \otimes p_1^*\mathscr{E}(n))$$
for any coherent, locally free sheaf $\mathscr{E}$ that are isomorphisms for $n \gg 0$.

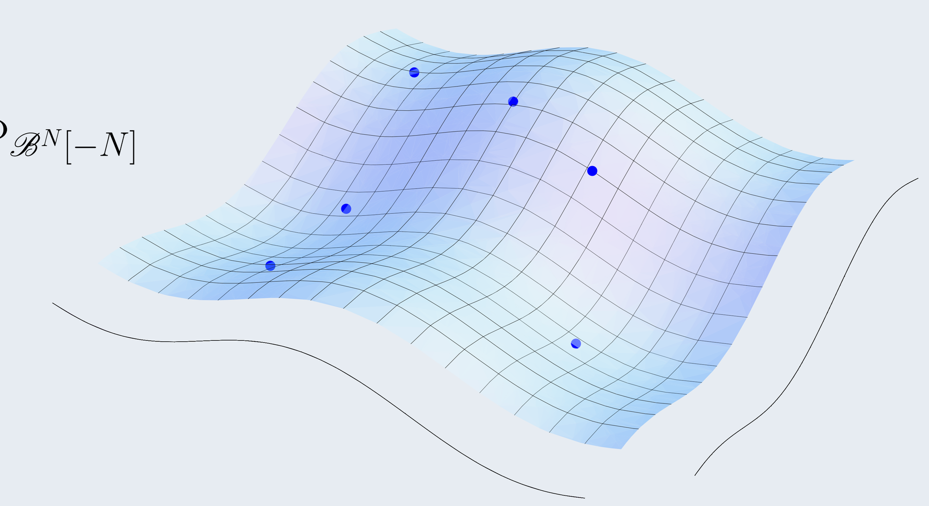
- By a theorem of Orlov in [1], F is a **bounded** functor, i.e. for all $\mathscr{F} \in \mathrm{Coh}(X)$ we have $\mathscr{H}^i(F(\mathscr{F})) = 0$ for $i \notin [M_1, N_1]$: this is why we will only find a finite number of sheaves $\mathscr{B}^i$
- Why these are the right sheaves: if the functor is actually a FM transform $\Phi_{\mathscr{B}}$ for some $\mathscr{B} \in D^b_{Coh}(X \times Y)$, then for $n \gg 0$ we have
$$\begin{aligned}\mathscr{H}^i(\Phi(\mathscr{E}(n))) &= \mathscr{H}^i(Rp_{2*}(\mathscr{B} \overset{L}{\otimes} p_1^*\mathscr{E}(n)))\\ &= p_{2*}(\mathscr{H}^i(\mathscr{B} \overset{L}{\otimes} p_1^*\mathscr{E}(n)))\\ &= p_{2*}(\mathscr{H}^i(\mathscr{B}) \overset{L}{\otimes} p_1^*\mathscr{E}(n))\end{aligned}$$

When can we find an isomorphism?

Here's one case in which we can prove it:

Theorem Let X and Y be smooth projective varieties with $\dim(X) = 1$. Consider an exact functor
$$F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$$
such that the sheaves $\mathscr{B}^i$ as computed above satisfy
- $\mathscr{B}^i = 0$ for $i \neq N$
- $\mathscr{B}^N = \oplus k(p_i, q_i)$

Then F is isomorphic to $\Phi_{\mathscr{B}^N[-N]}$



Note that functors of this type are not full nor faithful in general.

Other results: All is well at the generic point

Theorem Let X be a smooth projective variety over a field k, K a finite separable field extension of k with $\mathrm{trdeg}_k K \leq 1$ or K purely transcendental of transcendence degree 2. Consider a functor
$$H : D^b_{Coh}(X) \to \mathrm{Vect}_K$$
contravariant, cohomological, finite type. Then H is representable by an object $A \in D^b_{Coh}(X_K)$, i.e. for every $C \in D^b_{Coh}(X)$ we have
$$H(C) = \mathrm{Hom}_{D^b_{Coh}(X_K)}(j^*C, A)$$
where $j : X_K \to X$ is the base change morphism.

- A **cohomological functor** $H : D^b_{Coh}(X) \to \mathrm{Vect}_k$ is a functor that sends a triangle
$$X \to Y \to Z \to X[1] \to \ldots$$
to a long exact sequence
$$\ldots \to H(X) \to H(Y) \to H(Z) \to H(X[1]) \to \ldots$$
- A cohomological functor is said to be **of finite type** if
$$\sum_i \dim(H(C[i])) < \infty$$
for all $C \in D^b_{Coh}(X)$.

Corollary Let X and Y be smooth projective varieties over k, with $\dim Y \leq 1$ or Y a rational surface. Consider an exact functor
$$F : D^b_{Coh}(X) \to D^b_{Coh}(Y)$$
Let $i : \eta \to Y$ denote the inclusion of the generic point of Y. Then there exists an $E \in D^b_{Coh}(X \times Y)$ such that
$$i^* \circ F \cong i^* \circ \Phi_E$$

- The proof uses ideas from the paper of Bondal and Van den Bergh [2], as well as base change techniques from [4]
- We don't know how to extend this isomorphism over an open set of Y.

References

[1] D. Orlov, *Equivalences of Derived Categories and K3 Surfaces*
[2] A. Bondal, M. Van den Bergh, *Generators and Representability of Functors in Commutative and Noncommutative Geometry*
[3] A. Canonaco, P. Stellari, *Twisted Fourier-Mukai Functors*
[4] W. Lowen, M. Van den Bergh, *Deformation Theory of Abelian Categories*