Linear growth for curves on unnodal Enriques surfaces

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Problem

An Enriques surface S is a regular algebraic surface with K_S nontrivial but $2K_S \sim 0$. Given a linear system $|L|$ on S, such that $L^2 \geq 4$, we are interested in the g^1_ds on general curves in $|L|$. Here we want to see if the linear growth condition holds for these curves, that is,

$$\dim W^1_d(C) = d - \text{gon}(C),$$

$$\text{for gon}(C) \leq d \leq g - \text{gon}(C) + 2$$

and C general in $|L|$.

Background

Enriques surfaces are closely related to K3 surfaces, but yet they behave differently from K3 surfaces in many ways. In particular, given a base component free complete linear system, unlike on K3 surfaces, it is not the case that the Clifford index or the gonality of smooth curves in a linear system is constant (see [1]). However, we still expect that some of the theories on K3 surfaces could be extended to Enriques surfaces. In the paper of Aprodu and Farkas [2], it is proved that the linear growth condition is satisfied for smooth curves of gonality $\leq (g+2)/2$. We would like to see to what extent we can get a similar result for smooth curves on Enriques surfaces. Here we can only present our work in progress, which gives a partial linear growth.

Vector bundle techniques

Let S be an Enriques surface and C a smooth curve with $C^2 \geq 0$. Let $|A|$ be a base point free linear system on a curve C with $h^0(C, A) = 2$. Then we have a vector bundle $\mathcal{F}_{C,A}$ and its dual, $\mathcal{E}_{C,A}$, associated to the pair (C, A), and which are given by the exact sequence

$$0 \to \mathcal{F}_{C,A} \to H^0(A) \otimes \mathcal{O}_s \to A \to 0,$$

and which dualized gives us

$$0 \to H^0(A)^\vee \otimes \mathcal{O}_S \to \mathcal{E}_{C,A} \to N_{C|S} \otimes A^\vee \to 0.$$

From the exact sequences we can easily deduce the following properties of these two vector bundles:

1. $\mathcal{E}_{C,A}$ is generated by its global sections away from finite points.
2. $H^0(\mathcal{F}_{C,A}) = H^1(\mathcal{F}_{C,A}) = 0$, $H^2(\mathcal{E}_{C,A}) = 0$
3. $c_1(\mathcal{E}_{C,A}) = C$, $c_2(\mathcal{E}_{C,A}) = \deg A$

We refer to [3] for more details.

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Main result

Theorem Let S be an unnodal Enriques surface and $|L|$ a linear system on S such that the general curves $C \in |L|$ are smooth and non-exceptional with gonality $\text{gon}(C) \geq 3$. Then $\dim W^1_d(C) = d - \text{gon}(C)$ for $d \leq g - \text{gon}(C)$.

Idea of the proof

Let S and L be as above. We only need to prove that $\dim W^1_d(C) \leq d - \text{gon}\,(C)$, since the dimension is always $\geq d - \text{gon}(C)$ for any curve C. We want to consider components of the scheme $\mathcal{W}^1_d(U)$ which parametrizes pairs (C, A), where $A \in W^1_d(C)$ and U is open in $|L|$. Since $\dim |L| = g - 1$, the linear growth condition is equivalent to showing that $\dim \mathcal{W}^1_d(U) \leq g - 1 + d - k$, where k is the generic gonality of smooth curves in $|L|$.

- If the general A has base points in a component W of $W^1_d(C)$, then $\dim W \leq \dim W^1_{d-1}(C) + 1$. The reason is that for each A' in $W^1_{d-1}(C)$, we have that $|A' + P|$ is a g^1_d with base-point P for general $P \in C$.

  It follows that we only need to consider components of $W^1_d(C)$ where the general g^1_d is base-point free. This enables us to consider vector-bundles $\mathcal{E}_{C,A}$ associated to the general pairs (C, A).

- If $\mathcal{E}_{C,A}$ is simple and μ_L-stable for general A in a component W of $W^1_d(C)$, then it can be shown that $\dim W \leq \rho(g, 1, d) + 2$, where $\rho(g, 1, d) := -g + 2d - 2$. This implies that $\dim W \leq d - \text{gon}(C)$ whenever $d \leq g - \text{gon}(C)$.

  It follows that we can assume that $\mathcal{E}_{C,A}$ is either non-simple or non-stable for (C, A) general in its component.

- Since $\mathcal{E}_{C,A}$ is either non-simple or non-stable, it can be shown that $\mathcal{E}_{C,A}$ sits inside an exact sequence

  $$0 \to M \to \mathcal{E}_{C,A} \to N \otimes \mathcal{I}_\xi \to 0,$$

  where M and N are line-bundles satisfying $h^0(S, M) \geq 2$, $h^0(S, N) \geq 2$, $M \geq N$, $|N|$ base-component free, and where ξ is a zero-dimensional subscheme of length $d - M.N$. Note that we must have $M + N \sim C$.

  It is then shown that the dimension of possible extensions of M and $N \otimes \mathcal{I}_\xi$ for various ξ of length $d - M.N$ is small enough that linear growth is satisfied.

- It is shown that the dimension of these extensions is $3(d - M.N) + h^1(S, \mathcal{O}_S(M - N)) - 1$.

  For each such vector-bundle $\mathcal{E}$, there are $2g - 2d + 3 - h^0(S, \mathcal{E} \otimes \mathcal{E}^*)$ dimensions of pairs (C, A) such that $\mathcal{E}_{C,A} = \mathcal{E}$.

  Our challenge has been to find an estimate for $h^0(S, \mathcal{E} \otimes \mathcal{E}^*)$ and $M.N$.

- It appears that whenever $\mathcal{E}$ is non-simple or non-stable, it follows that $h^0(S, \mathcal{E} \otimes \mathcal{E}^*) \geq h^0(S, \mathcal{O}_S(M - N))$, and that when the general curves C in $|L|$ are non-exceptional, $M.N \geq \text{gon}(C) - 1$. (This is where we assume that S is unnodal.)

  It is then concluded that $\dim \mathcal{W}^1_d(U) \leq g - 1 + d - k$ (where k is the generic gonality in $|L|$), as desired.

Why it is interesting

In [4, Statement T, page 280], it is conjectured that if $\dim W^1_{\text{gon}(C)}(C) = 0$, then linear growth is satisfied for $d \leq g - \text{gon}(C) + 2$. As far as we know, no counter-example is known.
Aprodu has proved that if linear growth is satisfied for $d \leq g - \text{gon}(C) + 2$, then the Green and Green–Lazarsfeld conjectures are satisfied. The Green conjecture states that the Clifford index can be read off the minimal free resolution of the canonical ring of C, and the Green–Lazarsfeld conjecture states that the gonality can be read off the minimal free resolution of the graded ring of a line bundle A for $\deg(A) >> 0$.

References

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