Universal Torsors and Manin's Conjecture

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Manin's conjecture

Let k be a number field, X a Fano variety over k such that $X(k)$ is dense in X. Let $H : X(k) \to \mathbb{R}_{\geq 0}$ be a height function associated to the anticanonical bundle of X as in [1].

Example 1. If $k = \mathbb{Q}$ and X is anticanonically embedded in some projective space $\mathbb{P}^n$. Given $x \in X(k)$ the height function is defined by $H(x) = \sup_{i=0,\dots,n} |x_i|$, where $x_0, \dots, x_n \in \mathbb{Z}$ such that $x = (x_0 : \cdots : x_n)$ and $\gcd(x_0, \dots, x_n) = 1$.

Question 1. Is there an open subset $U \subset X$ and a positive constant C such that

$$\#\{x \in U(k) : H(x) \leq B\} \sim CB(\log B)^{\mathrm{rk}\,\mathrm{Pic}(X)-1}$$

as $B \to +\infty$?

Aim

I am working to develop geometric tools and a method that unifies the already known proofs of special cases of Manin's conjecture via universal torsors.

Setting

Let k be a number field and X a proper variety over k that has an integral model $\mathfrak{X}$ over the ring of integers $\mathcal{O}_k$ of k. Under base extension we have that $\mathfrak{X}(\mathcal{O}_k) = X(K)$. Let $Cl(\mathcal{O}_k)$ be the ideal class group of $\mathcal{O}_k$. Given an $\mathfrak{X}$-torsor Y under $\mathbb{G}^r_{m,\mathfrak{X}}$, we can describe $\mathfrak{X}(\mathcal{O}_k)$ in terms of $\mathcal{O}_k$-points on the twists of Y as disjoint union $\mathfrak{X}(\mathcal{O}_k) = \bigsqcup_{a \in Cl(\mathcal{O}_k)^r} \pi_a({}_aY(\mathcal{O}_k))$.

Result 1. If Y is a locally closed subset of some affine $\mathcal{O}_k$-space then we can give a good description of its twists, i.e. ${}_aY(\mathcal{O}_k)$ is set of lattice points with affine coordinates that satisfy some equations and some coprimality conditions.

At this point one can use the parametrization induced by the twisted torsors to count points of bounded height.

Question 2. In which cases do we have such a good situation and how does this help in counting rational points of bounded height?

Universal torsors

If X is smooth and split, then X has exactly one universal torsor Y [2], which can be realized as an open subset of $A = \mathrm{Spec}(\mathrm{Cox}(X))$ and the Cox ring $\mathrm{Cox}(X)$ of X is a finitely generated k-algebra. So, if the universal torsor construction is defined over $\mathcal{O}_k$ we can apply Result 1.

Question 3. Under which hypothesis does $\mathfrak{X}$ admit a universal torsor? The answer involves representability and étale locally constance of the relative Picard functor and descent theory.

If L is an ample line bundle on X, then the ideal defining the complement of Y in A is generated by $H^0(X, L)$, [3]. In order to describe the embedding $Y \to A$ in terms of equations we need to express global sections of ample line bundles as elements of $\mathrm{Cox}(X)$.

Example 2. Toric varieties. Let X be a complete smooth toric variety, then $Cox(X) = k[x_1, \dots, x_n]$ is a polynomial ring, the ideal I is generated by monomials and the universal torsor Y of X is defined over $\mathcal{O}_k$. If k is an imaginary quadratic number field and the anticanonical sheaf of X is generated by global sections, then we prove Manin's conjecture counting lattice points in bounded domains on the twisted universal torsors over $\mathcal{O}_k$.

Example 3. Let X be the cubic fourfold in $\mathbb{P}^5$ defined by $x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2 = 0$, $k = \mathbb{Q}$. The universal torsor of X is an open subset of a hypersurface in $\mathbb{A}^{10}$, it is defined over $\mathbb{Z}$ and gives a parametrization of the points on X that allows to prove Manin's conjecture [4].

References

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