Del Pezzo Surfaces in Weighted Projective Spaces

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We study singular del Pezzo surfaces that are quasi-smooth and well-formed weighted hypersurfaces. We give an algorithm how to classify all of them.

Classically, Fano varieties were assumed to be smooth. However, during the last decades the progress in the area — including the development of the Minimal Model Program in the works of Birkar, Corti, Hacon, Kawamata, McKernan, Mori, Shokurov and others — both gave tools and posed problems dealing with mildly singular Fano varieties. Unfortunately, singular Fano varieties do not form a bounded family even in dimension two, and their classification seems to be absolutely hopeless in higher-dimensions. Nevertheless, we know many partial classification-type results about singular del Pezzo surfaces thanks to combined efforts of many algebraic geometers.

The existence of an orbifold Kähler–Einstein metric on a Fano orbifold X is equivalent to the existence of a solution to the global complex Monge–Ampere equation on X. This problem remains out of reach even in dimension two. We know many obstructions to the existence of such an orbifold metric. However, del Pezzo surfaces with quotient singularities are very far from being classified. Mostly because of this, the Calabi problem for them, determining when they admit a Kähler–Einstein metric, is very far from being solved. So, it seems natural to impose more restrictions on the class of singular del Pezzo surfaces under consideration, e.g. to consider only singular del Pezzo surfaces that are quasi-smooth and well-formed hypersurfaces in weighted projective spaces.

The problem of finding all such hypersurfaces was posed by Orlov a long time ago in order to test his conjecture about the existence of a full exceptional collection on del Pezzo surfaces with quotient singularities.

It is seems unlikely that Orlov's problem has a *nice* solution for all I at once. However, the results by Cheltsov, Johnson, Kollár, and Shramov indicate it seems possible to solve it for any fixed I. The main purpose of this paper is to prove this and to give an algorithm that solves Orlov's Problem for any fixed I.

This paper provides the algorithm to find the answer to Orlov's problem for any fixed I, as well as the general form of the answer for any I. The surprisingly rigid form of the answer gives us the power of drawing conclusions about the hypersurfaces for all I at once, without explicitly calculating them. The program that calculates the classification of the hypersurfaces for any given I is available from the author. We hope that our classification can be useful to produce vast number of examples of non-Kähler–Einstein del Pezzo surfaces with quotient singularities.

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