An effective criterion for algebraic contractibility of rational curves

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Background

Let $\mathbf{E}$ be a (possibly reducible) curve on a non-singular complex algebraic surface $\mathbf{X}$.

Castelnuovo-Enriques (1901): $\mathbf{E}$ rational and $(\mathbf{E}, \mathbf{E}) = -1$ iff $\mathbf{E}$ *contracts* to a nonsingular (algebraic) surface $\mathbf{X'}$.

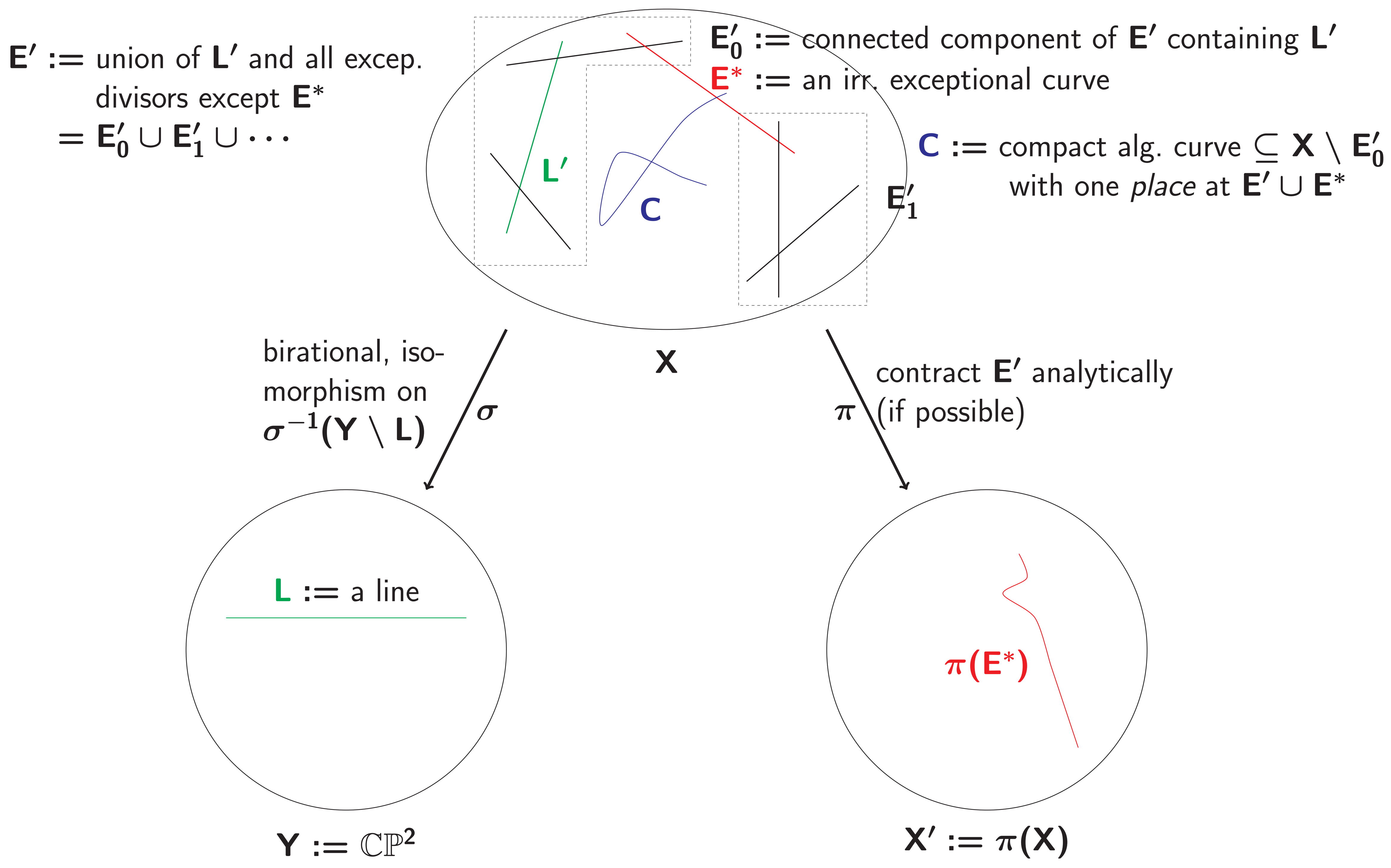
Grauert (1962): intersection matrix of $\mathbf{E}$ negative definite iff $\mathbf{E}$ can be contracted to an *analytic* surface $\mathbf{X'}$.

Question: How to determine if $\mathbf{X'}$ is algebraic?

Artin (1962): $\mathbf{X'}$ has rational singularity $\Rightarrow$ $\mathbf{X'}$ is algebraic. (Only previously known criterion which is *computable*.)

Other criteria: Morrow-Rossi(1975), Brenton(1977), Franco-Lascu(1999), Schröer(2000), Bădescu(2001), Palka(2012).

Main result (2013)



Question: When is $\mathbf{X'}$ *algebraic*?
Geometric answer: Iff there exists $\mathbf{C}$ as above.
Algebraic (effective) answer: Iff all the *key forms* of $\mathbf{E^*}$ are polynomials.

Applications

- Identifying $\mathbf{X'} \setminus \pi(\mathbf{E^*})$ with $\mathbb{C}^2$ gives a new correspondence:

  normal algebraic compactifications of $\mathbb{C}^2$ with one (irreducible) curve at infinity $\longleftrightarrow$ algebraic curves in $\mathbb{C}^2$ with one *place* at infinity

How deep is the correspondence? If the curve is rational, does it imply the surface has rational singularities?

- Results about normal analytic compactifications of $\mathbb{C}^2$ with one irreducible curve at infinity:
  - Singularities of the surface and the curve at infinity: geometric genus, multiplicity, etc.
  - Construction of non-algebraic normal Moishezon surfaces with *simplest* possible singularities.
  - Groups of automorphisms and moduli spaces.
  - Explicit equations of algebraic compactifications.
- New examples of divisors with non-finitely generated graded rings.
- Algorithm to determine if a valuation is *negative* or non-positive on $\mathbb{C}[\mathbf{x}, \mathbf{y}] \setminus \{\mathbf{0}\}$.

Key forms

- Global analogue of MacLane's (1936) *key polynomials* of valuations.
- Encode 'cancellations' of the order of vanishing along a divisor.
- May fail to be polynomials.

Examples of key forms

Let $\mathbb{C}^2 := \{[\mathbf{x} : \mathbf{y} : \mathbf{1}]\} \subseteq \mathbb{CP}^2$, $\mathbf{O} := [\mathbf{1} : \mathbf{0} : \mathbf{0}] \in \mathbf{L}_\infty$, $(\mathbf{u}, \mathbf{v}) := (\mathbf{1}/\mathbf{x}, \mathbf{y}/\mathbf{x})$ coordinates near $\mathbf{O}$.

σ_i	Exceptional divisor	key pols in $(\mathbf{u}, \mathbf{v})$-coordinates and orders of vanishing along $\mathbf{E_i}$	key forms in $(\mathbf{x}, \mathbf{y})$-coordinates and orders of pole along $\mathbf{E_i}$
Blow-up of $\mathbf{O}$	$\mathbf{E_1}$; $\mathbf{O_1} := (\frac{\mathbf{u}}{\mathbf{v}} = \mathbf{0}, \mathbf{v} = \mathbf{0})$	$\mathbf{u} \mapsto \mathbf{1}, \mathbf{v} \mapsto \mathbf{1}$	$\mathbf{x} \mapsto \mathbf{1}, \mathbf{y} \mapsto \mathbf{0}$.
Blow-up of $\mathbf{O_1}$	$\mathbf{E_1}$; $\mathbf{O_2} := (\frac{\mathbf{u}}{\mathbf{v}^2} = \mathbf{1}, \mathbf{v} = \mathbf{0})$; $\mathbf{E_2}$	$\mathbf{u} \mapsto \mathbf{2}, \mathbf{v} \mapsto \mathbf{1}$	$\mathbf{x} \mapsto \mathbf{2}, \mathbf{y} \mapsto \mathbf{1}$.
Blow-up of $\mathbf{O_2}$	$\mathbf{E_1}$; $\mathbf{E_2}$; $\mathbf{E_3}$	$\mathbf{u} \mapsto \mathbf{2}, \mathbf{v} \mapsto \mathbf{1}, \mathbf{u} - \mathbf{v}^2 \mapsto \mathbf{3}$	$\mathbf{x} \mapsto \mathbf{2}, \mathbf{y} \mapsto \mathbf{1}, \mathbf{x} - \mathbf{y}^2 \mapsto \mathbf{1}$.

Idea of the proof (of the main result)

- Identify $\mathbf{X'} \setminus \pi(\mathbf{E^*})$ with $\mathbb{C}^2$. Let $\mathbf{g_1}, \ldots, \mathbf{g_n}$ be the key forms of the order of pole δ along $\mathbf{E^*}$.
- Easy part: if all $\mathbf{g_j}$'s are polynomials, then $\mathbf{g} := (\mathbf{1}, \mathbf{g_1}, \ldots, \mathbf{g_n}) : \mathbb{C}^2 \hookrightarrow \mathbf{W}\mathbb{P}$ induces an embedding of $\mathbf{X'}$ into $\mathbf{W}\mathbb{P}$, where $\mathbf{W}\mathbb{P} := \mathbb{P}^n(\mathbf{1}, \delta(\mathbf{g_1}), \ldots, \delta(\mathbf{g_n}))$.
- Hard part: if one of the $\mathbf{g_j}$'s is not a polynomial, then $\mathbf{X'}$ cannot be algebraic. Uses the fact that if $\mathbf{X'}$ is algebraic, then there would be an algebraic curve $\mathbf{C'} \subseteq \mathbf{X'}$ such that $\mathbf{C'}$ does not intersect $\pi(\mathbf{E'})$. The proof follows from a careful comparison of *degree-wise Puiseux* factorization of $\mathbf{g_j}$'s with that of the defining polynomial of $\mathbf{C'}$.

Remark: δ is a *degree-like function*, and the proof is motivated by the theory of compactifications of affine varieties determined by degree-like functions.

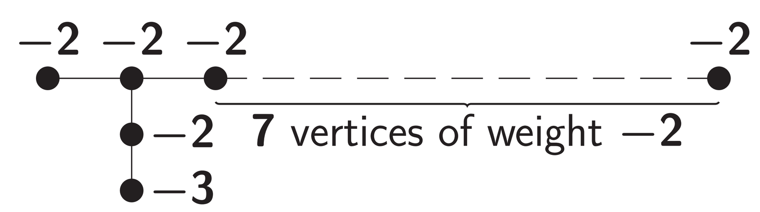
Example

Choose affine coordinates $(\mathbf{u}, \mathbf{v})$ on $\mathbb{CP}^2$ such that $\mathbf{L} = \{\mathbf{u} = \mathbf{0}\}$.

Let $\mathbf{C_j} := \{\mathbf{f_j} = \mathbf{0}\}$ for $\mathbf{j} = \mathbf{1}, \mathbf{2}$, where $\mathbf{f_1} := \mathbf{v}^5 - \mathbf{u}^3$ and $\mathbf{f_2} := (\mathbf{v} - \mathbf{u}^2)^5 - \mathbf{u}^3$.

Let $\mathbf{X_j}$, $\mathbf{j} = \mathbf{1}, \mathbf{2}$, be the surface obtained by resolving the singularity of $\mathbf{C_j}$ at $(\mathbf{0}, \mathbf{0})$ and then successively blowing up $\mathbf{8}$ times the point of intersection of (strict transform of) $\mathbf{C_j}$ with the exceptional divisor.

Let $\mathbf{E^*_j}$, $\mathbf{j} = \mathbf{1}, \mathbf{2}$, be the last exceptional curve on $\mathbf{X_j}$. $\mathbf{C_1}$ and $\mathbf{C_2}$ are equisingular $\Rightarrow$ $\mathbf{X_1}$ and $\mathbf{X_2}$ are diffeomorphic.



$\mathbf{X'_1}$ and $\mathbf{X'_2}$ have a unique singular point with the same dual graph of resolution.

W.r.t. coordinates $(\mathbf{x}, \mathbf{y}) := (\mathbf{1}/\mathbf{u}, \mathbf{v}/\mathbf{u})$ (of the complement $\mathbb{CP}^2 \setminus \mathbf{L}$) the key forms of $\mathbf{E^*_1}$ are $\mathbf{x}, \mathbf{y}, \mathbf{y}^2 - \mathbf{x}^5$, and the key forms of $\mathbf{E^*_2}$ are $\mathbf{x}, \mathbf{y}, \mathbf{y}^2 - \mathbf{x}^5, \mathbf{y}^2 - \mathbf{x}^5 - 4\mathbf{x}^{-1}\mathbf{y}^4$.

It follows that $\mathbf{X'_1}$ is algebraic, whereas $\mathbf{X'_2}$ is *not*.