NEW EXAMPLES OF LAGRANGIAN FIBRATIONS

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Lagrangian Fibrations

A PROJECTIVE IRREDUCIBLE SYMPLECTIC VARIETY (ISV for short) is a simply connected complex algebraic variety X, whose $H^{2,0}$ is generated by an everywhere nondegenerate form σ. A lagrangian fibration (LF for short) is an ISV X together with a surjective morphism $f : X \to B$, with connected fibers, such that the generic fiber is a lagrangian subvariety of X. These are natural generalizations to higher dimensions respectively of K3 surfaces and elliptic K3 fibrations. By Arnold-Liouville theory, the generic fiber of a LF is an abelian variety. LFs form an important class of ISVs thanks to the following:

Theorem 1. *(Matsushita) A surjective morphism $f : X \to B$ with connected fibers from an ISV X to a projective manifold B with $0 < \dim B < \dim X$, is a LF (hence in particular $\dim B = \frac{1}{2}\dim X$).*

Moreover, as it can be easily checked in the K3 case, the LF structure imposes a strong condition on the base:

Theorem 2. *(Hwang [2]) Let $X \to B$ be a LF with B smooth, then $B \cong \mathbb{P}^n$.*

It remains open the problem of characterizing LFs with singular bases.

LFs from K3 Fano flags

MY research focuses on the study of a particular class of examples of LFs over non compact bases (the definition is the same as before but with X and B quasi-projective instead of projective). The aim is to extend such LFs over (possible singular) compactifications of the base, using natural degenerations of Jacobians or Prym varieties over singular curves.

This class of examples is given by the relative Jacobian of K3-Fano flags. A K3-Fano flag is a pair $S \subset V$ where V is a Fano 3-fold and $S \in |-K_V|$ is a K3 surface. Fixing S, we have a universal family $\mathcal{V} \to B$ of K3-Fano flags, not compact in general. Considering the intermediate Jacobian of each 3-fold we obtain the relative Jacobian $\mathcal{J} \to B$.

Theorem 3. *(Markushevich [3]) The relative Jacobian $\pi : \mathcal{J} \to B$ of the universal family $\mathcal{V} \to B$ is a LF.*

Using the well known classification of Fano 3-folds of Mori and Mukai [4], we are reduced to study a finite number of such relative Jacobians. The 4 dimensional fibrations arising in this setting have already been studied by Bouali and Markushevich. Hence the following interesting case is in dimension 6.

It is the LF corresponding to a K3 surface S given by the intersection of two generic divisors of type $(1,1,1)$ and $(1,1,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$. Hence the corresponding Fano 3-folds are of type $(1,1,2)$, and the base of the LF is an open subset of $\mathbb{P}^3$. Moreover each V has two conic bundle structures, respectively over $\mathbb{P}^1 \times \mathbb{P}^1$ and over $\mathbb{P}^2$. By a result of Beauville [1], the intermediate Jacobians of such Fano 3-folds can be identified via the conic bundles respectively with the Prym variety of an étale double cover and the Jacobian of a genus 3 curve. We have determined B and we are currently trying to compactify it using degenerations of Jacobians and Prym varieties.

References

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