Topological Bounds for Minimal Threefolds

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Introduction

In [1], Paolo Cascini and De-Qi Zhang provide a bound for the singularities of a threefold X depending only on the topology of its resolution, in particular depending on the Picard number $\rho(X)$.

Proposition 0.1. *Let X be a smooth projective threefold and assume that*

$$X = X_0 \dashrightarrow \ldots \dashrightarrow X_k = Y$$

is a sequence of steps for the K_X*-minimal model program of X. Then,*

$$\Xi(Y) \leq 2\rho(X).$$

Question

Is it possible to find a bound for the number of minimal models of a threefold X depending only on the topology of X?

References

[1] P. Cascini, D. Q. Zhang (2012) *Effective finite generation for adjoint rings*, arXiv:1203.5204

Bounding the number of divisorial contractions

The strategy is to bound the number of steps, $\mathcal{S}$, and then bound the number of operations at each step, $\mathcal{B}_{\mathcal{S}}$.

$$\text{Number of Minimal Models} = (\mathcal{B}_{\mathcal{S}})^{\mathcal{S}}$$

The number of possible steps is given by the sum of the possible divisorial and small contractions. The number of divisorial contractions is clearly bounded by $\rho(X)$. By results in [1] the total number of flips is bounded by the total number of divisorial contractions. The number of steps, therefore, is bounded by $2\rho(X)$.

Given a divisor E, this can be contracted to a point or to a curves. In the first case an obvious bound is given by the Picard number of X. In the second case $\rho(X)$ controls the number of divisors but each of them can be contracted in two different ways. Let us assume that E is a divisor contracted to a curve C

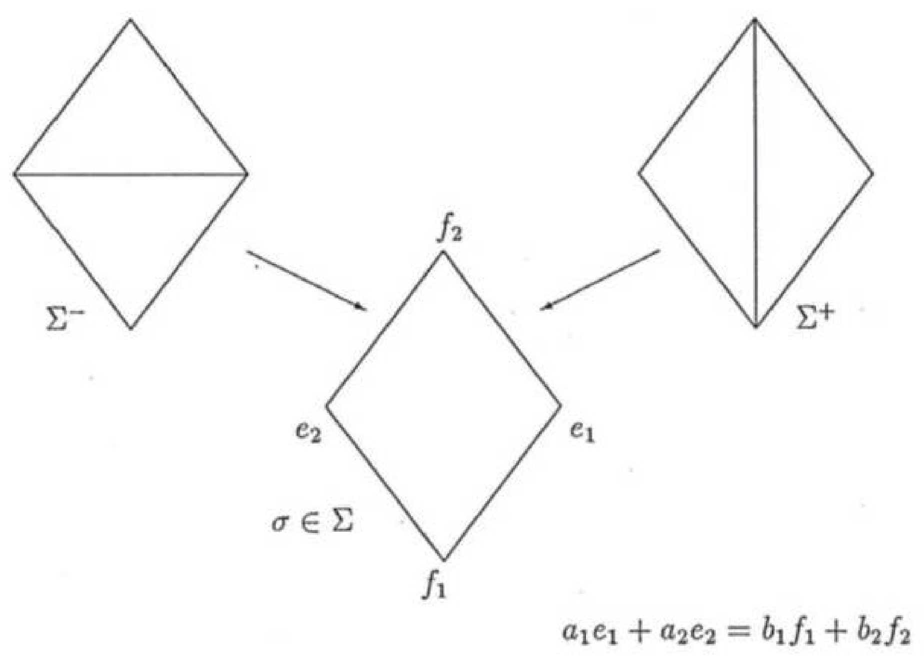
$$\pi : E \longrightarrow C \qquad (1)$$

π is a $\mathbb{P}^1$- bundle over C. Since the Picard number of a curve is always 1 and $\rho(E)$ can drops only by 1 through π, we conclude that $\rho(E) = 2$. Thus the cone of curves is 2 dimensional and there are only two extremal rays. Thus there are only two possible rulings. The number of divisorial contractions to a curve is, therefore, $2\rho(X)$.

Thus, in the worst case there are at most $3\rho(X)$ divisorial contractions.

Bounding the number of flips

To count the number of flips, we can fix a singular point and look at the local picture. How many are the flipping curves passing through one fixed singular point P?



This is the toric flip diagram $(a_1, a_2, -b_1, -b_2)$. These kind of diagram are classified by Danilov: the only ones that can occur are of the form $(a, 1, -b_1, -b_2)$ where $a > b_1$, $hcf(a, b_1) = 1$ and either $b_2 = 1$ or $a = b_1 + b_2$.

Fixing the singularity: only one flipping curve.

Conjecture: The hope is that the singular points of the variety determine, at least analytically, the curves that can be flipped. Studying the analytic expression of the singularities and the action that produces that singular point it is possible to find a finite number of preferential directions for the tangent line of the flipping curve. After we fix a tanget direction we conjecture that there is only one flipping curve passing through the singular point with that direction.