On the Chevalley's Conjecture

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Introduction

Our work is motivated by the conjecture set by Chevalley a few years before 1957:

Conjecture (Chevalley)**.** If X is a normal complete algebraic variety such that for every finite set $S \subset X$, there is an affine open subset $U \subset X$ such that $S \subset U$ then X is projective.

For an algebraic variety X we will use the following notions:

Definitions.
$a(X) := \sup\{n \mid$ every set of n points in X is contained in some open affine subset of $X\}$
$mqos(X)$ denotes the cardinality of $MQOS(X)$, the set of maximal quasiprojective open subsets of X
$\mathrm{CaDiv}(X)$ is the group of Cartier divisors, $F_1(X)$ the group of 1-cycles. By $\mathrm{CaDiv}(X)^\tau$ and $F_1(X)^\tau$ we denote the subgroups of numerically trivial Cartier divisors and 1-cycles. The intersection form gives us a perfect pairing between the spaces $N^1(X) := \mathrm{CaDiv}(X)/\mathrm{CaDiv}^\tau(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_1(X) := F_1(X)/F_1^\tau(X) \otimes_{\mathbb{Z}} \mathbb{R}$.
$\rho(X)$ – the Picard number of X, i.e. the dimension of $N^1(X)$ and $N_1(X)$.

The first result concerning Chevalley's conjecture was given in 1966 by Kleiman in [4]:

Theorem (Kleiman)**.** *If X is a $\mathbb{Q}$-factorial complete algebraic variety and $a(X) \geq 2\rho(X)$ then X is projective.*

Kleiman's method was improved in 1996 by Kollár who exhibited in [5] a simple way of showing that it is enough to assume $a(X) \geq \rho(X) + 1$.
In 1999 Włodarczyk [6] introduced a new method of approaching the problem. He noticed that if an algebraic variety X satisfies $a(X) \geq mqos(X)$ then X must be quasiprojective (and thus $a(X) = \infty$ and $mqos(X) = 1$). Then he showed that if X is a complete algebraic variety satisfying some technical condition (satisfied by $\mathbb{Q}$-factorial varieties and toric varieties) then $MQOS(X)$ is finite, thus if X additionally satisfies $a(X) = \infty$ then X is projective.

Recently Benoist, [1], has shown that $MQOS(X)$ is finite for every normal algebraic variety thus proving:

Theorem (Benoist-Włodarczyk)**.** *If X is a normal algebraic variety satisfying $a(X) = \infty$ then X is quasiprojective.*

Our results

In [2] we have shown that normality is essential:

Theorem. *For every $n \geq 2$ there exists an n-dimensional nonprojective variety satisfying $a(X) = \infty$ (and thus also $mqos(X) = \infty$).*

We have used Włodarczyk's result to enhance Kleiman's proof obtaining:

Theorem. *If X is a $\mathbb{Q}$-factorial complete algebraic variety and $a(X) \geq \rho(X)$ then X is projective.*

On the other hand, in [3] we have obtained a result, which shows that the theorem above is almost sharp:

Theorem. *For every integer $n \geq 2$ there exists a smooth complete threefold X with $\rho(X) = n+1$ and $a(X) = n-1$.*

In [3] we established the result which shows that for normal varieties one cannot obtain a projectivity criterion based on $a(X)$ and $\rho(X)$.

Theorem. *For every $t \geq 5$ there exists a complete normal toric threefold X with $a(X) = t$ and $\rho(X) = 0$.*

Further research

Despite the recent breakthroughs there are still some open questions:

Question. Let X be a variety smooth in codimension 2. Does $MQOS(X)$ need to be finite? If additionally $a(X) = \infty$ does X need to be projective?

Question. Does there exist a $\mathbb{Q}$-factorial complete variety with $a(X) = \rho(X) - 1$?

References

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