Seshadri constants of line bundles on hyperelliptic surfaces

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Introduction

First let us recall some definitions.

Definition. Let X be a smooth projective variety and let L be a nef line bundle on X. Let $x \in X$.
The Seshadri constant of L at x is defined as

$$\varepsilon(L, x) = \inf \left\{ \frac{LC}{\operatorname{mult}_x C} : \, x \in C \right\},$$

where the infimum is taken over all irreducible curves $C \subset X$ passing through x.

Definition. The global Seshadri constant of L at x is defined as

$$\varepsilon(L) = \inf_{x \in X} \varepsilon(L, x).$$

Definition. A hyperelliptic surface, sometimes called bielliptic, is a minimal surface whose Kodaira dimension equals zero and irregularity equals one.

Alternatively, a surface S is hyperelliptic if $S \cong (A \times B)/G$, where A and B are elliptic curves, and G is a finite group such that A/G is an elliptic curve and $B/G \cong \mathbb{P}^1$.
Every hyperelliptic surface is a special cases of a quasi-bundle, i.e. fibration from surface to a smooth curve such that the smooth fibers are isomorphic and singular fibers are multiples of smooth curves.

Hyperelliptic surfaces were classified by G. Bagnera and M. de Franchis at the beginning of XX century. There are seven types of these surfaces.
In 1990 F. Serrano in [Se] computed a basis of the group of classes of numerically equivalent divisors $\operatorname{Num}(S)$ for each of the hyperelliptic surface's type and the multiplicities of the singular fibres in each case. The result is the following:

Type	G	$m_1, \ldots, m_s$	Basis of $\operatorname{Num}(S)$
1	$\mathbb{Z}_2$	$2, 2, 2, 2$	$A/2, B$
2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$2, 2, 2, 2$	$A/2, B/2$
3	$\mathbb{Z}_4$	$2, 4, 4$	$A/4, B$
4	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$2, 4, 4$	$A/4, B/2$
5	$\mathbb{Z}_3$	$3, 3, 3$	$A/3, B$
6	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$3, 3, 3$	$A/3, B/3$
7	$\mathbb{Z}_6$	$2, 3, 6$	$A/6, B$

We say that L is a line bundle of type (α, β) on a hyperelliptic surface if $L \equiv \alpha \cdot A/\mu + \beta \cdot \mu/\gamma B$, where $\mu = \gcd(m_i)$ and γ is the order of the group G.
If $L_1 \equiv (\alpha_1, \beta_1)$, $L_2 \equiv (\alpha_2, \beta_2)$ then $L_1.L_2 = \alpha_1\beta_2 + \beta_1\alpha_2$.

Main result

Theorem. *Let S be a hyperelliptic surface. Let L be a line bundle of type $(1, 1)$ on S. Then*

$$\varepsilon(L) = 1.$$

Sketch of the proof. Let $L \equiv (1, 1)$. Of course L is a nef line bundle.
We show that for each hyperelliptic surface type and for each point $p \in S$ we have $\varepsilon(L, p) \geq 1$.
We have to consider several cases for each hyperelliptic surface's type, depending on the position of point x. We separately compute or estimate the Seshadri constant of L at points lying on singular fibres A/μ, separately at points on singular fibres $m_i A/\mu$ if $m_i/\mu < 1$, and separately at points lying out of the singular fibres.
For a reduced irreducible curve $C \equiv (\alpha, \beta)$ we have to estimate the fraction $\frac{LC}{m}$. We intersect L with divisors of the basis of $\operatorname{Num}(S)$ and use Bezout's Theorem to get lower bounds on α and β.
On hyperelliptic surfaces of type 4 and 6, Bezout's Theorem is not enough. But we can use the genus formula and Hodge Index Theorem to get the assertion. □

Further research

The aim of our project is to provide new theorems on positivity of the line bundles on hyperelliptic surfaces and their blow-ups.
We would like to compute exact values or estimate Seshadri constants of line bundles of type (α, β) at one point, and as a consequence we will give estimation of global Seshadri constants of line bundles on hyperelliptic surfaces. We give estimation of multiple point Seshadri constants on hyperelliptic surfaces as well. We examine when the line bundle $L = \pi^* L_S - k \sum_{i=1}^{r} E_i$ on the blow-up of hyperelliptic surface is k-very ample. We intend to prove, using vanishing theorems, that on hyperelliptic surfaces the line bundle of type $(k+2, k+2)$ generates k-jets.

References

[Se] F. Serrano, *Divisors on Bielliptic Surfaces and Embeddings in* $\mathbb{P}^4$, Math. Z. 203 (1990), 527-533.