The specialization index

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Problem

Take R henselian DVR, Frac $R = K$, residue field k with $\overline{k} = k$, X variety over K.

When is $X(K) \neq \emptyset$?

Invariants

Index $i(X)$: gcd of degree of closed points
ν-invariant $\nu(X)$: minimum of degrees of closed points

Properties:

- $i(X) \mid \nu(X)$
- $X(K) \neq \emptyset \Leftrightarrow \nu(X) = 1$
- $\nu(X)$ harder to compute then $i(X)$!

Known results

Theorem (Graber, Harris, Starr). *Every rationally connected variety over the function field of a curve over $\mathbb{C}$ has a rational point.*

Theorem (Colliot-Thélène & Voisin, Esnault & Wittenberg, Nicaise).
Let X/K be proper, smooth and geometrically connected, char $k = 0$.
If $H^i(X, \mathcal{O}_X) = 0$ $\forall i > 0$, then $i(X) = 1$.

Proposition (Lafon). *There exists a proper, smooth and geometrically connected variety X over $\mathbb{C}((t))$ such that $H^i(X, \mathcal{O}_X) = 0$ $\forall i > 0$ and $X(K) = \emptyset$.*

References

[1] Jean-Louis Colliot-Thélene and Claire Voisin. Cohomologie non ramifiée et conjecture de Hodge entiere. *Duke Mathematical Journal*, 161(5):735–801, 2012.

[2] Tom Graber, Joe Harris, and Jason Starr. Families of rationally connected varieties. *Journal of the American Mathematical Society*, 16(1):57–67, 2003.

[3] Guillaume Lafon. Une surface d'Enriques sans point sur $\mathbb{C}((t))$. *Comptes Rendus Mathematique*, 338(1):51–54, 2004.

Specialization index

Definition. Take $\mathcal{X}/R$ proper, then $\mathcal{D}_{\mathcal{X}}$ is the set of integers $d > 0$ s.t. $\exists$ zero devisor of degree d on $\mathcal{X}_K$ with support in $\mathrm{sp}_{\mathcal{X}}^{-1}(x)$ for some $x \in \mathcal{X}(k)$.
The specialization index $i_{sp}(\mathcal{X})$ is the minimum of $\mathcal{D}_{\mathcal{X}}$.

Take X/K proper. A model for X is a proper and flat scheme $\mathcal{X}$ over R with an isomorphism $\mathcal{X}_K \to X$. We set

$$\mathcal{D}_X = \cap\{\mathcal{D}_{\mathcal{X}} \mid \mathcal{X} \text{ is an } R\text{-model for } X\}.$$

The **specialization index** $i_{sp}(X)$ is the minimum of $\mathcal{D}_X$.

Properties

Properties:

- $i(X) \mid i_{sp}(X)$
- $i_{sp}(X) \leq \nu(X)$

The specialization index is a stronger invariant then the index:

Proposition. *There exists a smooth, proper, geometrically connected K-curve C with $i(C) = 1$ and $i_{sp}(C) > 1$.*

So $i_{sp}(X)$ is intermediate between $i(X)$ and the existence of a rational point.

Result

Theorem.
Let X/K be smooth, proper and geometrically connected with char $k = 0$.
If $H^i(X, \mathcal{O}_X) = 0$ $\forall i > 0$, then $i_{sp}(X) = 1$.

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