Tropical points of multiplicity m

Nikita Kalinin, "Newton polygon of a planar singular curve", arXiv:1306.4688
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Tropical setup

Tropical semi-ring: $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$.
Field of Puiseux series: $\mathbb{K} = \{\sum_{\alpha \in I} c_\alpha t^\alpha | c_\alpha \in (\mathbb{C}^*), I \subset \mathbb{R}\}$, where t is a formal variable and I is a well-ordered set, i.e. each of its nonempty subsets has a least element.
Valuation map $\mathrm{val}: \mathbb{K} \to \mathbb{T}$ is defined as $\mathrm{val}(\sum_{\alpha \in I} c_\alpha t^\alpha) := -\min_{\alpha \in I} \alpha$ and $\mathrm{val}(0) := -\infty$.
Tropicalization $\mathrm{Trop}(V) \subset \mathbb{T}^n$ of an algebraic variety $V \subset \mathbb{K}^n$ is the image of V under the map val applied coordinate-wise.

Throughout the poster we consider a finite set $\mathcal{A} \subset \mathbb{Z}^2$ and a curve C given by the equation

$$F(x,y) = \sum_{(i,j) \in \mathcal{A}} a_{ij} x^i y^j = 0, a_{ij} \in \mathbb{K}$$

A point p is of **multiplicity m for C** if $\frac{\partial^{i+j}}{\partial^i x \partial^j y} F(x,y)_{|_p} = 0$ for each $0 \le i, j; i+j \le m-1$.

The **Newton polygon** of C is the set of all integer points in the convex hull of $\mathcal{A}$ in $\mathbb{R}^2$. The **extended Newton polygon** $\tilde{\mathcal{A}}$ of C is the convex hull of the set $\{((i,j),s) \in \mathbb{R}^2 \times \mathbb{R} | (i,j) \in \mathcal{A}, s \le \mathrm{val}(a_{ij})\}$. The projection $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ defines a **subdivision of the Newton polygon** by images of the faces of $\tilde{\mathcal{A}}$. Look at Figures 1,2 which illustrate the example below.

This subdivision is dual to $\mathrm{Trop}(C)$: each vertex V of $\mathrm{Trop}(C)$ corresponds to a face $\boldsymbol{d(V)}$ of the subdivision; each edge E of $\mathrm{Trop}(C)$ corresponds to an edge $\boldsymbol{d(E)}$ in the subdivision, the direction of the edge $d(E)$ is perpendicular to the direction of E.

The tropical curve $\mathrm{Trop}(C)$ equals to the set of non-smooth points of the piece-wise linear function $\mathrm{Trop}(F) = \max_{(i,j) \in \mathcal{A}} (ix + jy + \mathrm{val}(a_{ij}))$, i.e. to the set of points $(x,y) \in \mathbb{T}^2$ where the maximum is attained at least twice.

For a point $A \in \mathrm{Trop}(C)$ denote by $\boldsymbol{\mathrm{dep}(A)}$ the all vertices B of a tropical curve, such that B lies on a long edge passing through A.

Example. Consider a curve C defined by the equation $F(x,y) = t^{-3}xy^3 - (3t^{-3} + t^{-2})xy^2 + (3t^{-3} + 2t^{-2} - 2t^{-1})xy - (t^{-3} + t^{-2} - 2t^{-1} - 3)x + + t^{-2}x^2y^2 - (2t^{-2} - t^{-1})x^2y + (t^{-2} - t^{-1} - 3)x^2 + t^{-1}y - (t^{-1}+1) + x^3 = 0$
The point $(1,1)$ is of multiplicity 3 for C, the point $(0,0)$ is of multiplicity 3 for the curve $\mathrm{Trop}(C)$ (Figure 2) defined as the set of all non-smooth points of the function $\mathrm{Trop}(F(x,y)) =$

$$\max(3+x+3y, 3+x+2y, 3+x+y, 3+x, 2+2x+2y, 2+2x+y, 2+2x, y+1, 1, 3x)$$

The vertices of the tropical curve C have coordinates $(-2,0), (1,0), (2,0)$.

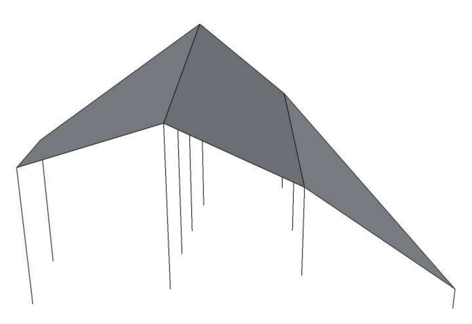

Figure 1: The extended Newton polygon

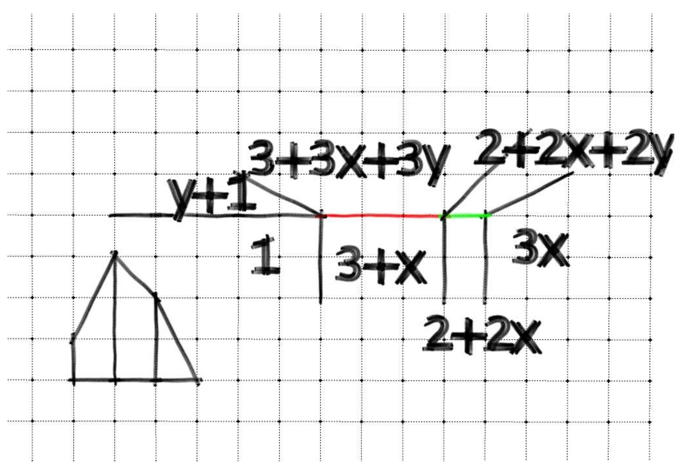

Figure 2: Subdivision of the Newton polygon, the tropical curve Trop(C), the function Trop(F)

Results

Consider a curve C defined in setup with the point $(1,1)$ of multiplicity m. Note that $\mathrm{Trop}((1,1)) = A = (0,0) \in \mathbb{T}^2$.

Main technical theorem. Suppose that the point $A \in A_l A_{l+1}$ lies on a maximal long edge $A_1 A_2 A_3 \dots A_{n-1} A_n$ and the point A **is not** a vertex of $\mathrm{Trop}(C)$ while A_i are vertices of $\mathrm{Trop}(C)$. Then the following is true:

- a) the edge $d(A_l A_{l+1})$ has length at least m
- b) the sum $S(A)$ of areas of the faces $d(A_1), d(A_2), \dots, d(A_n)$ corresponding to $A_1, A_2 \dots A_n$ is at least $m^2/2$.

For A — a vertex of C — we introduce

$$S(A) = \mathrm{area}(d(A)) + \sum_{B \in \mathrm{dep}(A)} \mathrm{area}(d(B)), \bar{S}(A) = \mathrm{area}(d(A)) + \frac{1}{2} \sum_{B \in \mathrm{dep}(A)} \mathrm{area}(d(B))$$

From the theorem above $S(A) \ge m^2/2$ if A is in an edge.
Exertion Theorem. If A is a vertex of C then $S(A) \ge \frac{3}{8}m^2$ and $\bar{S}(A) \ge m^2/4$.

Examples of sets $\bigcup_{B \in \mathrm{dep}(A)} d(B)$ are shown on Figures 3,4. The number $S(A)$ is the area of a such a set .

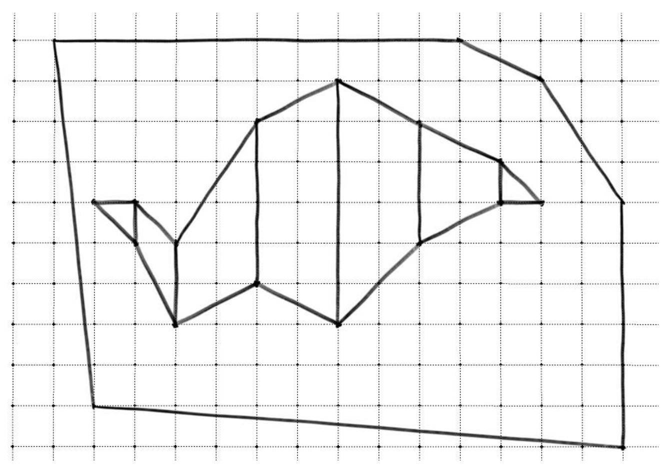

Figure 3: Dual picture, A is in an edge

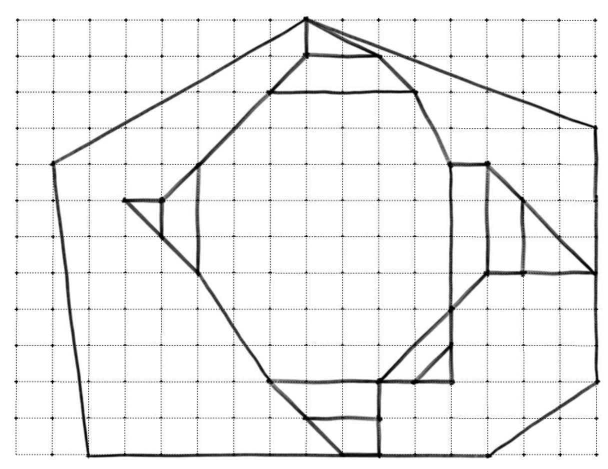

Figure 4: Dual picture, A is a vertex

Consider the set L of curves with given support $\mathcal{A}$ and such that the point $(1,1)$ of multiplicity m. It is clear that L is a linear space. This space L defines a matroid and we can construct the Bergman fan $T(m, \mathcal{A})$ of it.

M-coverability Theorem (in terms of Bergman fans). Each flag of flats $\varnothing \subsetneq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq \mathcal{A}$ corresponding to a vector in the fan $T(m, \mathcal{A})$ has the following property: for each $l \in \mathbb{N}$ the set $\mathcal{A} \setminus F_l$ is m-coverable.

What is essentially new?

I found **new necessary conditions** for a point to be realized as the tropicalization of a point of multiplicity m. These conditions are easily formulated in terms of Newton polygon subdivisions. Moreover, if one draws a curve through generic points with prescribed multiplicities then each face in the subdivision is under governorship of at most two singular points (Figure 5).
So, one can convert these results into a more precise definition of a tropical point of multiplicity m. Nevertheless an ambiguity is remained: if a singular point lies on the edge E then it is impossible to locate it unless $\omega_E(\bigcup_{B \in \mathrm{dep}(A)} d(B)) = m$.

Key notions and ideas of proofs

Let $B \subset \mathbb{Z}^2$ be a non-empty finite set. **Lattice width** of B in a direction $u \in \mathbb{Z}^2$ is the number $\omega_u(B) = \max_{x,y \in B} u \cdot (x-y)$. The minimal lattice width $\omega(B)$ is $\min_{u \in \mathbb{Z}^2} \omega_u(B)$.
Theorem. Suppose $\omega(B) > 0$. Then the area $S(B)$ of the polygon B satisfies the following inequality

$$S(B) \ge \frac{3}{8} \omega(B)^2$$

.
A finite set $B \subset \mathbb{Z}^2$ is called **m-coverable** if there is no set $\{l_i\}_{i=1..m-1}$ of lines such that cardinality $|B \setminus \bigcup l_i| = 1$.
Example. Two integer points give us a 1-coverable set while one point does not. The empty set is m-coverable for any m.

For $\mu \in \mathbb{R}$ denote by $\mathcal{A}_\mu$ the set $\{(i,j) \in \mathcal{A} | \mathrm{val}(a_{ij}) \ge \mu\}$.
Theorem. For each real number μ the set $\mathcal{A}_\mu$ is m-coverable.
Define the function $\hat{g}(x)$ to be the length of interval excised out of the line $z = g(x)$ by the graph of g.
Lemma. The length m_i of the edge $d(A_i A_{i+1})$ is not less than $m - \hat{g}(x_i)$.

Pictures

Lemma. Suppose g is convex on the interval $[a,b]$ and $g(a) = g(b)$. Then

$$\int_a^b \hat{g}(x)dx = (b-a)^2/2$$

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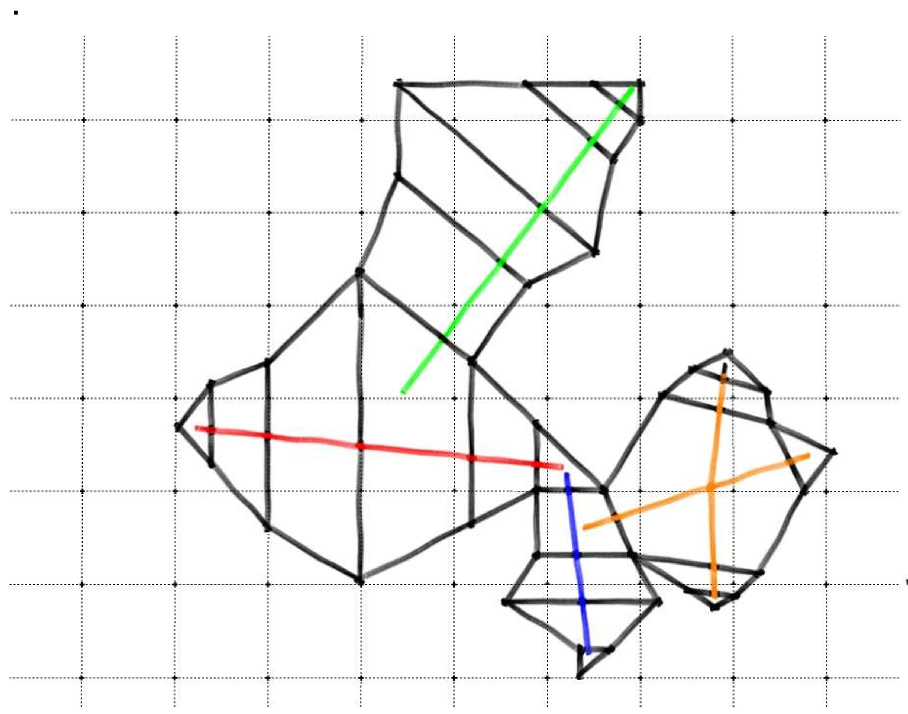

Figure 5: Governorships of singular points.

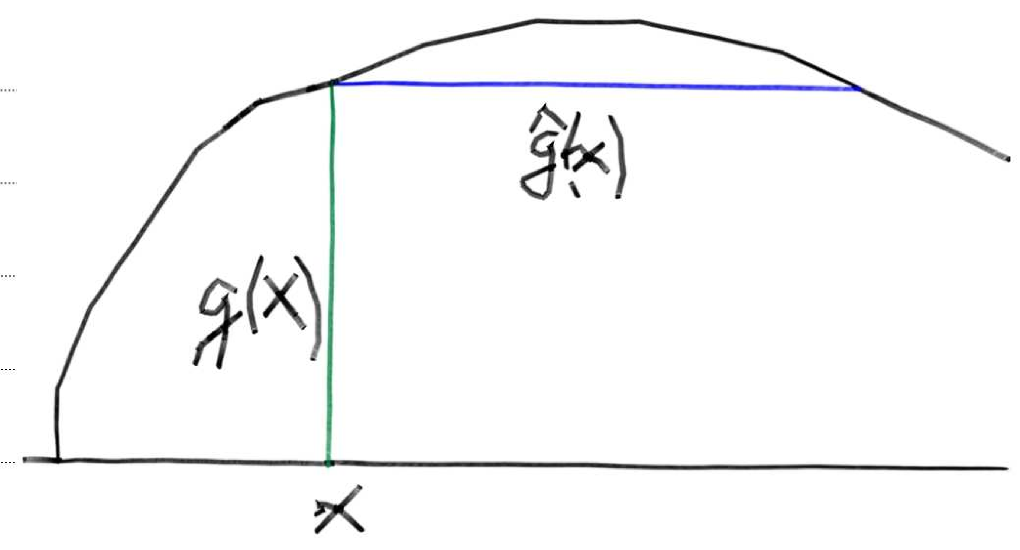

Figure 6: $\hat{g}(x) =$ length of the blue line.

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