Modifications of the Simpson moduli spaces of 1-dimensional sheaves on a projective plane by vector bundles

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1. Motivation

THE moduli spaces of vector bundles (in different settings) are non-compact, so one looks for compactifications. At least two approaches are possible.

1 **A rather standard approach:** compactifications by coherent sheaves. The objects on "the boundary" are supported on schemes of the same type as the vector bundles from the initial moduli problem.

2 **Another approach:** compactify the initial moduli spaces of vector bundles by locally free sheaves. This could only be possible if one allows the support to vary.

Simpson moduli spaces and the first approach

Theorem (C. Simpson)**.** *For an arbitrary smooth projective variety X and for an arbitrary numerical polynomial P there is a coarse projective moduli space $M := M_P(X)$ of semi-stable sheaves on X with Hilbert polynomial P.*

IN general M contains a closed subvariety M' of sheaves (we call them *singular*) that are not locally free on their support. Its complement M_B is an open dense subset whose points are sheaves that are locally free on their support. So, one could consider M as a compactification of M_B.

IF X is a surface and the Hilbert polynomial is linear, then the sheaves from M are supported on curves and the corresponding Simpson moduli space may be seen as a compactification of a certain moduli space of vector bundles on curves in X by torsion-free sheaves (on support).

Conclusion. *Simpson moduli spaces can be seen as examples of the first approach.*

Problem. *One of the unsatisfactory facts about Simpson compactifications is that M' does not have the minimal codimension (is not a divisor). Loosely speaking, one glues together too many different directions at infinity.*

2. Second approach: examples.

THE second approach in the case of 1-dimensional sheaves on a projective plane has been conducted in certain particular cases.

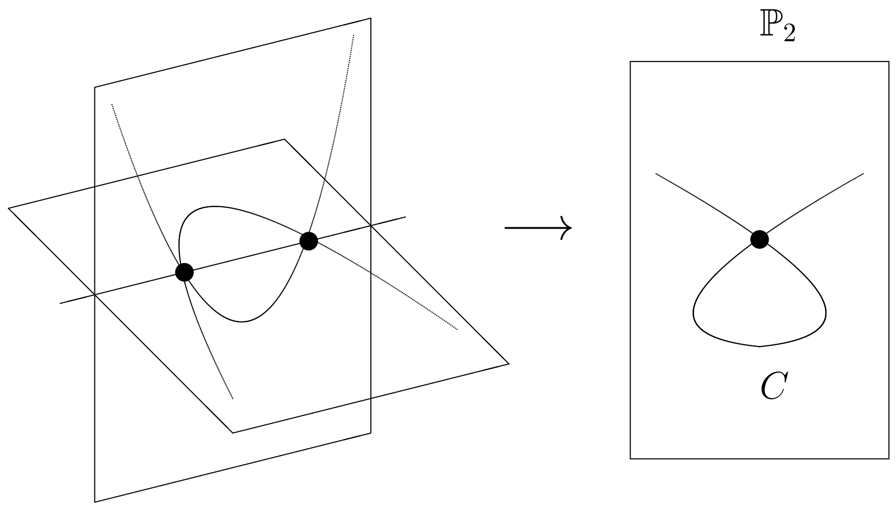
2.1 Ideals of points on curves

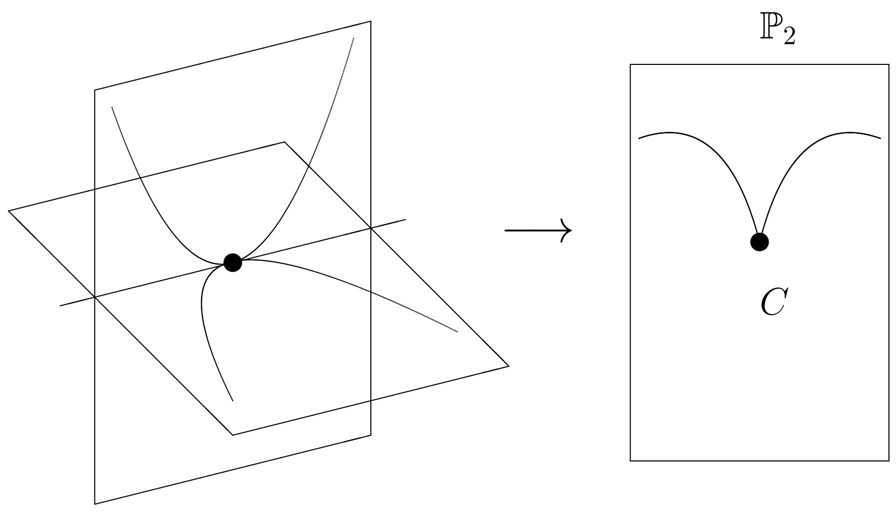
LET C be a plane curve and let p be its singular point, then the ideal sheaf of p on C and its dual, the only non-trivial extension

$$0 \to \mathcal{O}_C \to \mathcal{F} \to \Bbbk_p \to 0,$$

are singular stable sheaves and constitute a closed subvariety M'' in M' in the corresponding Simpson moduli spaces on $\mathbb{P}_2$.

IN [2] we describe a construction that substitutes every such sheaf by a variety of vector bundles on curves embedded in the reduced surface $D(p) := \mathbf{Proj}\,(\Bbbk_p \oplus \mathcal{I}_{\{p\}})$ as shown below.





THE surfaces $D(p)$ are flat degenerations of $\mathbb{P}_2$, the new sheaves (R-bundles) are obtained as flat limits of non-singular sheaves from M.

2.2 The case of cubic curves

IN the case of plane cubic curves C, i. e., for $M = M_{3m\pm1}(\mathbb{P}_2)$, we have $M'' = M'$ and the construction mentioned above provides an example of the second approach (see [1]).

Theorem. *1) The blow-up $\widetilde{M}$ of M at M' can be seen as a construction which substitutes the singular sheaves by R-bundles.*

2) Let $\mathcal{M}$ be the initial Simpson moduli problem. Then there exists another moduli problem $\widetilde{\mathcal{M}}$ over $\mathcal{M}$ such that $\widetilde{M}$ corepresents $\widetilde{\mathcal{M}}$ and the natural transformation $\widetilde{\mathcal{M}} \to \mathcal{M}$ is compatible with the blow-up $\widetilde{M} \to M$.

3. General aim

THE aim is to extend the approach for $M_{3m+1}(\mathbb{P}_2)$ to the general situation: modify the Simpson moduli spaces and the underlying moduli problems in order to obtain compactifications of the second type.

4. Geometry of M'.

TO do this we study the geometry of the boundary M'. For certain Hilbert polynomials a generic sheaf $\mathcal{E}$ in M is either an ideal sheaf of a zero-dimensional subscheme Z on a curve $C \subseteq \mathbb{P}_2$ or an extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_Z \to 0.$$

In this case $\mathcal{E}$ can only be singular if Z contains a singular point of C. This leads to the conclusion $\operatorname{codim}_M M' \geqslant 2$.

IN [3] we show that the equality holds for $M_{4m\pm1}(\mathbb{P}_2)$ and expect this to be true in general.

5. Work in progress, future plans

QUESTIONS to be answered include the following list.

1. Codimension of M'.
2. Irreducibility of M' or a description of its irreducible components.
3. Smoothness or a description of the singularity types of M'.

References

[1] O. Iena and G. Trautmann, *Modification of the Simpson moduli space $M_{3m+1}(\mathbb{P}_2)$ by vector bundles (I)*, ArXiv e-prints (2010).

[2] Oleksandr Iena, *Universal plane curve and moduli spaces of 1-dimensional coherent sheaves*, ArXiv e-prints (2011).

[3] ______, *On the singular sheaves in the fine Simpson moduli spaces of 1-dimensional sheaves supported on plane quartics*, ArXiv e-prints (2013).