# Hodge Theory and Derived Intersection

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## Hodge Theory

Let $X$ be a smooth projective variety over $\mathbb{C}$. Then

$$H^n_{dR}(X) = \oplus_{p+q=n} H^p(X, \Omega^q),$$

where the algebraic de Rham cohomology $H_{dR}(X)$ is the hypercohomology of the algebraic de Rham complex

$$\Omega^\bullet_{dR}(X) = 0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \to \dots$$

However, this theorem is stated algebraically, the first purely algebraic proof was given only in 1987 by Deligne and Illusie ([3]).

## Proof by Deligne and Illusie

First, assume that $X$ is a smooth scheme over a perfect field $k$ of characteristic $p > 0$ for a large enough $p$. Then consider the Frobenius twist $X'$, with relative Frobenius morphism $F : X \to X'$. The key step is to show that in this setting

- $F_*\Omega^\bullet_{dR} = 0 \to F_*\mathcal{O}_X \xrightarrow{d} \dots \xrightarrow{d} F_*\Omega^n_X \to 0$
- $\mathrm{Sym}(\Omega^1_{X'}[-1]) = 0 \to \mathcal{O}_{X'} \xrightarrow{0} \dots \xrightarrow{0} \Omega^n_{X'} \to 0$

are quasi-isomorphic complexes of $\mathcal{O}_{X'}$-modules, if the following two conditions hold

- $p > n = \dim(X)$,
- $X$ lifts to $W_2(k)$, i.e, there exists a smooth scheme $\tilde{X}$ over $W_2(k)$ with an isomorphism $\tilde{X} \times_{\mathrm{Spec}\, W_2(k)} \mathrm{Spec}\, k \xrightarrow{\sim} X$.

Then, by taking hypercohomology, we obtain the Hodge Decomposition in positive characteristics. For a smooth projective variety over $\mathbb{C}$, the statement follows by replacing $\mathbb{C}$ by a finitely generated $\mathbb{Z}$-algebra and reducing modulo a large prime $p$.

## Barannikov-Kontsevich Theorem ([2])

Let $X$ be a smooth quasi-projective variety over $\mathbb{C}$ equipped with a proper map $f : X \to \mathbb{A}^1$.The differentials $\Omega^i_X \xrightarrow{d+\wedge df} \Omega^{i+1}_X$ and $\Omega^i_X \xrightarrow{\wedge df} \Omega^{i+1}_X$ give rise to two complexes defined as

$$\Omega^\bullet_{X,d+\wedge df}: \quad 0 \to \mathcal{O}_X \xrightarrow{d+\wedge df} \Omega^1_X \xrightarrow{d+\wedge df} \dots \to 0$$

and

$$\Omega^\bullet_{X,\wedge df}: \quad 0 \to \mathcal{O}_X \xrightarrow{\wedge df} \Omega^1_X \xrightarrow{\wedge df} \dots \xrightarrow{\wedge df} \Omega^n_X \to 0.$$

The Barannikov-Kontsevich Theorem states that the hypercohomology spaces of the two complexes above are of the same finite dimension. In this case, the first purely algebraic proof of the above theorem was given by Ogus and Vologodsky ([5]) in 2007. Note that by taking $f = 0$, we obtain the Hodge Decomposition.

## Derived (self-)intersections

**Set-up**: $i : X \hookrightarrow S$, $j : Y \hookrightarrow S$ are closed embeddings of smooth schemes over a field $k$ intersecting cleanly but not necessarily transversely.

**Question**: under what circumstances is the derived intersection as simple as possible, or in other words whether $\mathbf{L}j^* i_* \mathcal{O}_X$ is formal (isomorphic to $\oplus \mathcal{H}^m(\mathbf{L}j^* i_* \mathcal{O}_X)[m]$ in $D^b(X)$)? In general, when is $\mathbf{L}j^* i_* V$ formal for vector bundles $V$?

Local calculation shows that the cohomology sheaves of $\mathbf{L}j^* i_* V$ are the same as of $q_*(p^* V \otimes \mathrm{Sym}(E^\vee[1]))$, where $E$ denotes the excess bundle, $p$ and $q$ denote the embeddings of $W$ into $X$ and $Y$ respectively.

## Theorem of Arinkin and Căldăraru ([1])

In the case of $X = Y$, $\mathbf{L}i^* i_* V$ is formal if and only if the following conditions hold:

- $k$ is of characteristic 0 or bigger than the codimension of $X$ in $S$,
- both the normal bundle $N_{X/S}$ of $X$ in $S$ and $V$ extend to the first infinitesimal neighborhood.

Mustaţă noted that the above conditions are very similar to the conditions appearing in the proof of Deligne and Illusie, and asked whether there exists an embedding $i$ and a vector bundle $F$ such that $\mathbf{L}i^* i_* V = (F_* \Omega^\bullet_{dR}(X))^\vee$.

## Construction

If $p > 0$, the ring of differential operators $D_X$ has nice properties:

- $D_X$ has a large center: $Z(D_X) \xrightarrow{\sim} \mathcal{O}_{\Omega^1_{X'}}$,
- $D_X$ is an Azumaya algebra over $Z(D_X)$ (the corresponding Azumaya algebra over $\Omega^1_{X'}$ will be denoted by $D$),
- the pullback of $D$ to $X'$ via the embedding to the zero section $X' \hookrightarrow \Omega^1_{X'}$ splits, $D|_{X'} = \mathcal{E}nd_{X'}(F_* \mathcal{O}_X)$,
- thus $(X', D|_{X'})$ and $(X', \mathcal{O}_{X'})$ are Morita equivalent.

## Theorem (Arinkin,Căldăraru,-)

*For $i : X' \hookrightarrow \Omega^1_{X'}$ and for the composite of maps $i_D : (X', \mathcal{O}_{X'}) \to (X', D|_{X'}) \to (\Omega^1_{X'}, D)$ we have*

- $\mathbf{L}i^* i_* \mathcal{O}_{X'} = (\mathit{Sym}(\Omega^1_{X'}[-1]))^\vee$,
- $\mathbf{L}i_D^* i_{D*} \mathcal{O}_{X'} = (F_* \Omega^\bullet_{dR}(X))^\vee$.

*Moreover, if $X$ lifts to $W_2(k)$ and $p > \dim(X)$, then the two complexes above are quasi-isomorphic by the Formality Theorem.*

**Remark:** This provides a new, purely algebraic proof of the Hodge Theorem.

## Derived intersection

Assume that $i : X \hookrightarrow S$ and the vector bundle $V$ satisfy the conditions above (char$k >$ codim$(X, S)$ or char$k = 0$; $V$ and $N_{X/S}$ extend to the first infinitesimal neighborhood). Let $j : Y \to S$ be a closed embedding of smooth schemes, such that the underived intersection $W = X \times_S Y$ is clean and it is a local complete intersection. Then

## Theorem (Arinkin,Căldăraru,-; and Grivaux ([4])

*With these assumptions $\mathbf{L}j^* i_* V = q_*(p^* V \otimes \mathit{Sym}(E^\vee[1])$ if and only if the short exact sequence $0 \to E^\vee \to N^\vee_{X/S} \to N^\vee_{W/Y} \to 0$ splits.*

## Construction for the twisted de Rham complexes

The proper map $X \to \mathbb{A}^1$ provides a proper map $f' : X' \to \mathbb{A}^1$. Let $j : X'_f \hookrightarrow \Omega^1_{X'}$ be the graph of $d(f')$. Let $j_D$ denote the composite of maps $(X'_f, \mathcal{O}_{X'_f}) \to (X'_f, D|_{X'_f}) \to (\Omega^1_{X'}, D)$.

## Theorem (Arinkin, Căldăraru,-)

*With the notations above*

- $\mathbf{L}j^* i_* \mathcal{O}_{X'} = (\Omega^\bullet_{X', d(f')})^\vee$,
- $\mathbf{L}j_D^* i_{D*} \mathcal{O}_{X'} = (F_* \Omega^\bullet_{X, d+\wedge d(f)})^\vee$.

*Moreover, if $\mathbf{L}j^* i_* \mathcal{O}_{X'}$ is formal, then the two complexes above are quasi-isomorphic, providing a new, purely algebraic proof in a special case of the Barannikov-Kontsevich Theorem.*

Remark: The above theorem only works in the case when $W$ is a local complete intersection.

## Questions

- What happens if $W$ is not a local complete intersection? In that case what complex should replace the excess bundle?
- In general assume we have closed embeddings of smooth schemes $i : X \to S$ and $j : Y \to S$, and a sheaf of Azumaya-algebras $\mathcal{A}$ on $S$, so that $\mathcal{A}|_X$ and $\mathcal{A}|_Y$ split. Let us consider similarly as before the maps $i_{\mathcal{A}}$ and $j_{\mathcal{A}}$. Under what condition are $Ext^*(i_* M, j_* N) = Ext^*(i_{\mathcal{A}} M, j_{\mathcal{A}} N)$ for all $i \geq 0$ for some vector bundles $M$ and $N$ on $X$ and $Y$ respectively?

## References


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