ZARISKI-DECOMPOSITIONS ON THREEFOLDS

GAeL XXI
KTH and Stockolm University,
Stockolm, June 24th-28th, 2013

Enrica Floris

IRMA, 7 rue René Descartes, 67000 Strasbourg
floris@math.unistra.fr

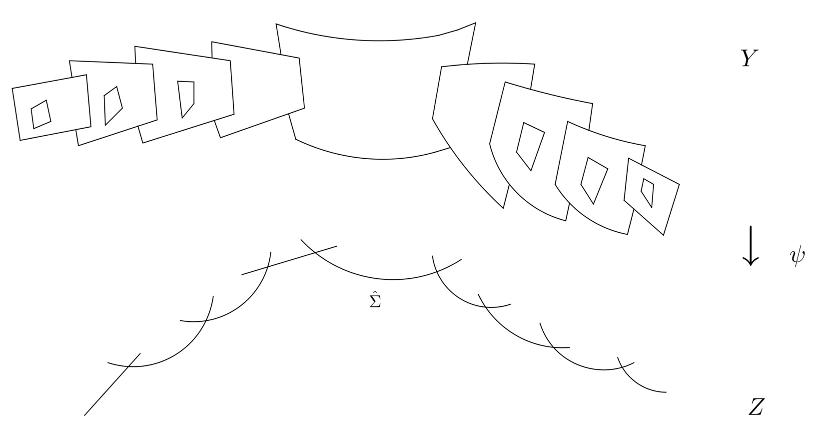
Zariski decomposition on surfaces Let S be a smooth projective surface defined over **C**. Let D be an effective divisor on S. Zariski proved in [8] the existence of two divisors P,N such that
(1) $N = \sum a_i N_i$ is effective, P is nef and D = P + N;
(2) either N = 0 or the matrix $(N_i \cdot N_j)$ is negative definite;
(3) $(P \cdot N_i) = 0$ for all i.
Such a decomposition is called Zariski decomposition of D and it is unique. Fujita in [4] generalized the statement to pseudoeffective divisors. Moreover he noticed in [5] that the divisor P is the unique divisor that satisfies the following property:
(α) for any birational model $f : X' \to X$ and any nef divisor L on X' such that $f_* L \leq D$ we have $f_* L \leq P$.

In higher dimension: the Fujita-Zarski decomposition.
There are several generalizations of the Zariski decomposition in higher dimension.
The property (α) gives rise to the following.
Definition Let X be a smooth complex projective variety and D a pseudoeffective divisor. A decomposition $D = P_f + N_f$ is called Fujita-Zariski decomposition if
(1) $N_f \geq 0$;
(2) P_f is nef;
(3) for any birational model $f : X' \to X$ and any nef divisor L on X' such that $f_* L \leq D$, we have $f_* L \leq P$.
It follows from the definition that, if a Fujita-Zariski decomposition exists, then it is unique.
The question of the existence of the Fujita-Zariski decomposition has been discussed for a longtime because of the importance of its role in applications in birational geometry: Birkar [1] and Birkar-Hu [2] proved the equivalence between the existence of a log minimal model for a pair and the existence of a Fujita-Zariski decomposition for the log canonical divisor.
First of all many examples show that, instead of looking for a decomposition of a pseudoeffective divisor D on X, it is more natural to look for a decomposition of f*D on a suitable birational model $f : Y \to X$. If such a decomposition exists, we say that D admits birationally a Fujita-Zariski decomposition. Even in this setting the existence of such a decomposition fails.
Indeed very recently J. Lesieutre in [6] found an example of a three-dimensional smooth variety X and a pseudoeffective divisor D on X such that its diminished base locus is dense in X and that cannot admit a Fujita-Zariski decomposition on any birational model of X.

The Nakayama-Zariski decomposition and the diminished base locus. The diminished base locus of a pseudoeffective divisor D, is defined as

$$\mathbf{B}_-(D) = \bigcap_{A \text{ ample}} \mathbf{B}(D+A)$$

where $\mathbf{B}(D+A) = \bigcap \{ \text{Supp}\Delta \mid \Delta \geq 0, \Delta \approx D+A \}$.
The diminished base locus depends only on the numerical equivalence class of D. A decomposition that is easier to use for computations and very imporant in applications is the Nakayama-Zariski decomposition.



We will not give a precise definition (see [Definition III.1.12, Definition III.1.16, 7]), we just list here the properties that we will need.

- $D = P_\sigma(D) + N_\sigma(D)$ with $P_\sigma(D)$ nef and $N_\sigma(D)$ effective.
- A Nakayama-Zariski decomposition of D is also a Fujita-Zariski decomposition of D.
- If X is smooth, the support of $N_\sigma(D)$ is the divisorial part of the diminished base locus.
- If D admits birationally a Nakayama-Zariski decomposition, then $\mathbf{B}_-(D)$ is closed.

From the above properties it follows that, if D admits birationally a Nakayama-Zariski decomposition, then $B_-(P_\sigma(D))$ is closed and of codimension at least 2.

For threefolds the converse is true:

Theorem ([3]) Let X be a complex projective smooth variety of dimension 3. Let D be a pseudoeffective divisor such that its diminished base locus is the union of a finite number of curves. Then there exists a birational morphism $\mu: Y \to X$ such that μ*D has a Nakayama-Zariski decomposition on Y.

Proof of the theorem. The idea of the proof is constructing the birational model $\mu: Y \to X$ and verifying that $P_\sigma(\mu^*D)$ is nef. For the sake of simplicity, we explain here the case where the diminished base locus is one connected smooth curve, $B_-(D)=\Sigma$, the proof in the general case being analogous. There are mainly two steps. The goal of the first step is making the conormal bundle of Σ semistable and of sufficiently high degree. This is achieved with a sequence of blow ups of smooth curves $\varphi: Z \to X$. The diminished base locus $B_-(\varphi^*D)$ is the union of the strict transform of Σ and of two chains of rational curves.
In the second step we blow up the curves of $B_-(\varphi^*D)$ in a suitable order, as shown in the picture.
We obtain $\psi: Y \to Z$ and we definethe morphism μ as $\psi \circ \varphi$. Finally, the divisor $N_\sigma(\mu^*D)$ is supported on the exceptional divisor of ψ. Thanks to a result by Nakayama we are able to compute all the coefficients of $N_\sigma(\mu^*D)$.
The locus where $P_\sigma(\mu^*D)=D- N_\sigma(\mu^*D)$ is not nef is contained in the support of $N_\sigma(\mu^*D)$ and since the components of $N_\sigma(\mu^*D)$ are birational to ruled surfaces the verification of the nefness of $P_\sigma(\mu^*D)$ can be done explicitely.

Bibliography

[1] C. Birkar, On existence of log minimal models and weak Zariski decompositions, Math Annalen, 354 no.2,(2012), 787-799.
[2] C. Birkar, Z. Hu, On log minimal models and Zariski decompositions, arXiv:1302.4015.
[3] E. Floris, On the Fujita-Zariski decomposition on threefolds, arXiv:1305.1441.
[4] T. Fujita, On Zariski problem, Proc. Japan Acad., Ser. A, 55, (1979), 106-110.
[5] T. Fujita, Zariski decomposition and canonical rings of elliptic threefolds, J. Math. Soc. Japan, 38, (1986), no. 1, 19-37.
[6]J. Lesieutre, The diminished base locus is not always closed. arXiv:1212.3738.
[7] N. Nakayama, Zariski-decomposition and abundance, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, (2004).
[8] O. Zariski, The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface, Ann. of Math., (2) 76, (1962), 560-615.