

Tango curves and the Effective Non-vanishing in positive characteristic

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The Effective Non-vanishing Conjecture (ENV)

Let X be a complete normal variety over an algebraically closed field k and let B be an effective \mathbb{R} -divisor on X such that (X, B) is klt. Let D be a nef Cartier divisor on X and assume that $D - (K_X + B)$ is nef and big. Then

$$H^0(X, D) \neq 0.$$

Char 0 case

Although the ENV is highly open in arbitrary dimension, Kawamata proved the following.

Theorem ([K00]). ENV holds in char 0 if $\dim X = 2$.

The key step consists in proving that, given a fibration f of X on a smooth curve, the sheaf $f_*\mathcal{O}_X(D)$ is locally free and semipositive (**Logarithmic Semipositivity**).

Only partial results are known in higher dimension.

Question

Does ENV hold for surfaces in characteristic $p > 0$?

Char p case

Let X be a smooth algebraic surfaces over a field k of positive characteristic. The interest in the previous question arises from the failure of the Logarithmic Semipositivity in positive characteristic (some examples on ruled surfaces can be found in [X06]). In this setting, Xie has a partial result.

Theorem ([X06]). ENV holds in char $p > 0$ if $\dim X = 2$ except possibly in the following cases:

- X is a ruled surface with $h^1(\mathcal{O}_X) \geq 2$;
- X is a quasi-elliptic surface with $\chi(\mathcal{O}_X) < 0$;
- X is a surface of general type with $\chi(\mathcal{O}_X) < 0$.

Tango curves

Tango's work in positive characteristic focuses on curves C which verify the following condition:

$$n(C) := \max \left\{ \deg \left[\frac{(df)}{p} \right] \mid f \in K(C), f \notin K^p(C) \right\} > 0.$$

Such curves exist as the following example shows.

Example ([Tan72]). Let us consider the following smooth curve in characteristic 3:

$$\{x^3y + y^3z + z^3x = 0\} \subset \mathbb{P}^2.$$

Choosing $f = \frac{x-z}{y}$, one deduces that $n(C) = 1$.

Definition. A curve C is called a **pre-Tango curve** (resp. **Tango curve**) if $n(C) > 0$ (resp. $n(C) = (2g - 2)/p$).

An ample divisor L on C such that $L^p \leq (df)$ (resp. $L^p = (df)$) for some $f \in K(C) \setminus K(C)^p$ is called a pre-Tango structure (resp. Tango structure) on C .

The existence of pre-Tango structures on C is guaranteed if $g \geq p$.

“Funny” curves in ruled surfaces

The Frobenius morphism on a curve C gives the sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow F_*\mathcal{O}_C \rightarrow \mathcal{B}^1 \rightarrow 0.$$

Combining it with the existence of a pre-Tango structure L on C , we obtain a non-trivial extension $\mathcal{E} \in H^1(C, L^{-1})$ such that $F^*(\mathcal{E}) = \mathcal{O}_C \oplus L^p$.

This construction produces a curve $\Gamma \subset X := \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$, given by the section

$$H^0(C, \mathcal{O}_C) \hookrightarrow H^0(X, \mathcal{O}_X(p) \otimes \pi^*L^{-p}).$$

Singularity of Γ and ENV

The restriction of π to the “funny” curve Γ we constructed is purely inseparable and $(\Gamma \cdot F) = p$, where F is the general fibre of the ruled surface. Furthermore Γ is “usually” singular, as the following result shows.

Theorem ([Tak10]). Let C be a pre-Tango curve. Then Γ is smooth iff C is Tango.

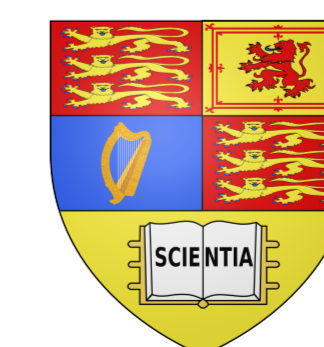
To be more precise, the number of singular points of Γ is at most $2g - 2 - pl$, where $l := \deg L$.

In order to study the ENV on non-minimal ruled surfaces, we start with a strictly pre-Tango structure on C and define the boundary $B := \varepsilon\Gamma$. If $g: X' \rightarrow X$ is the resolution of the singularities of Γ , we can define a new boundary $B' := g^*B - [\varepsilon p]\Sigma E_i$ on X' (E_i are the exceptional divisors).

So far we can prove that the existence of certain pre-Tango structures implies the existence of a 2-dimensional counterexample for ENV.

References

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