# Tango curves and the Effective Non-vanishing in positive characteristic

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## The Effective Non-vanishing Conjecture (ENV)

Let $X$ be a complete normal variety over an algebraically closed field $k$ and let $B$ be an effective $\mathbb{R}$-divisor on $X$ such that $(X, B)$ is klt. Let $D$ be a nef Cartier divisor on $X$ and assume that $D-(K_X+B)$ is nef and big. Then

$$H^0(X, D) \neq 0.$$

## Char 0 case

Although the ENV is highly open in arbitrary dimension, Kawamata proved the following.

**Theorem** ([K00])**.** ENV holds in char 0 if $\dim X = 2$.

The key step consists in proving that, given a fibration $f$ of $X$ on a smooth curve, the sheaf $f_*\mathcal{O}_X(D)$ is locally free and semipositive (**Logarithmic Semipositivity**).
Only partial results are known in higher dimension.

## Question

Does ENV hold for surfaces in characteristic $p > 0$?

## Char $p$ case

Let $X$ be a smooth algebraic surfaces over a field $k$ of positive characteristic. The interest in the previous question arises form the failure of the Logarithmic Semipositivity in positive characteristic (some examples on ruled surfaces can be found in [X06]). In this setting, Xie has a partial result.

**Theorem** ([X06])**.** ENV holds in char $p > 0$ if $\dim X = 2$ except possibly in the following cases:

- $X$ is a ruled surface with $h^1(\mathcal{O}_X) \geq 2$;
- $X$ is a quasi-elliptic surface with $\chi(\mathcal{O}_X) < 0$;
- $X$ is a surface of general type with $\chi(\mathcal{O}_X) < 0$.

## Tango curves

Tango's work in positive characteristic focuses on curves $C$ which verify the following condition:

$$n(C) := \max\left\{ \deg \left\lfloor \frac{(df)}{p} \right\rfloor \;\middle|\; f \in K(C),\ f \notin K^p(C) \right\} > 0.$$

Such curves exist as the following example shows.

**Example** ([Tan72])**.** Let us consider the following smooth curve in characteristic 3:

$$\{x^3y + y^3z + z^3x = 0\} \subset \mathbb{P}^2.$$

Choosing $f = \frac{x-z}{y}$, one deduces that $n(C) = 1$.

**Definition.** A curve $C$ is called a **pre-Tango curve** (resp. **Tango curve**) if $n(C) > 0$ (resp. $n(C) = (2g-2)/p$).
An ample divisor $L$ on $C$ such that $L^p \leq (df)$ (resp. $L^p = (df)$) for some $f \in K(C)\backslash K(C)^p$ is called a pre-Tango structure (resp. Tango structure) on $C$.

The existence of pre-Tango structures on $C$ is guaranteed if $g \geq p$.

## "Funny" curves in ruled surfaces

The Frobenius morphism on a curve $C$ gives the sequence

$$0 \to \mathcal{O}_C \to F_*\mathcal{O}_C \to \mathcal{B}^1 \to 0.$$

Combining it with the existence of a pre-Tango structure $L$ on $C$, we obtain a non-trivial extension $\mathcal{E} \in H^1(C, L^{-1})$ such that $F^*(\mathcal{E}) = \mathcal{O}_C \oplus L^p$.
This construction produces a curve $\Gamma \subset X := \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$, given by the section

$$H^0(C, \mathcal{O}_C) \hookrightarrow H^0(X, \mathcal{O}_X(p) \otimes \pi^* L^{-p}).$$

## Singularity of $\Gamma$ and ENV

The restriction of $\pi$ to the "funny" curve $\Gamma$ we constructed is purely inseparable and $(\Gamma \cdot F) = p$, where $F$ is the general fibre of the ruled surface. Furthermore $\Gamma$ is "usually" singular, as the following result shows.

**Theorem** ([Tak10])**.** Let $C$ be a pre-Tango curve. Then $\Gamma$ is smooth iff $C$ is Tango.

To be more precise, the number of singular points of $\Gamma$ is at most $2g-2-pl$, where $l := \deg L$.
In order to study the ENV on non-minimal ruled surfaces, we start with a strictly pre-Tango structure on $C$ and define the boundary $B := \varepsilon\Gamma$.
If $g\colon X' \to X$ is the resolution of the singularities of $\Gamma$, we can define a new boundary $B' := g^*B - \lfloor \varepsilon p \rfloor \Sigma E_i$ on $X'$ ($E_i$ are the exceptional divisors).
So far we can prove that the existence of certain pre-Tango structures implies the existence of a 2-dimensional counterexample for ENV.

## References

[K00] Kawamata Y., *On effective non-vanishing and base-point-freeness*, Asian J. Math., 4 (2000), 173-182;

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