Irreducible symplectic manifolds and orthogonal modular forms

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Orthogonal modular varieties are a class of locally symmetric variety with a modular interpretation for certain irreducible symplectic manifolds. Differential forms on an orthogonal modular variety can be interpreted as modular forms for an orthogonal group and so arithmetic information about the modular forms can be used to obtain geometric information about the modular variety.

1 Irreducible Symplectic Manifolds

Definition 1.1 *A complex manifold X is called an irreducible symplectic manifold (hyperkähler manifold) if:*

1. *X is a compact Kähler manifold;*
2. *X is simply-connected;*
3. *$H^0(X, \Omega^2_X) \cong \mathbb{C}\omega$ where ω is an everywhere non-degenerate holomorphic 2-form*

There are many reasons to be interested in irreducible symplectic manifolds:

- Irreducible symplectic manifolds are a generalisation of the K3 surfaces (the other natural generalisation being the Calabi-Yau varieties).
- An important class of Ricci-flat manifolds are the compact Kähler manifolds with vanishing first chern class. By a theorem of Bogomolov, these decompose into a product of complex tori, Calabi-Yau varieties and irreducible symplectic manifolds.
- The moduli of Hyperkähler manifolds are significant in Quantum field theory.

Four classes of irreducible symplectic manifold have been discovered but it is not known if there are any more. The four classes are:

- *(irreducible symplectic manifolds of deformation $K3^{[n]}$-type)* The length-n Hilbert scheme $S^{[n]} = Hilb^n(S)$ for a $K3$ surface S or its deformations.
- *(generalised Kummer varieties)* Let A be a complex torus of dimension 2 and let $A^{[n+1]} = Hilb^{n+1}(A)$ with the morphism $p : A^{[n+1]} \to A$ given by addition. Then $X := p^{-1}(0)$ is an irreducible symplectic manifold.
- *(O'Grady's 6-dimensional example)* A 6-parameter deformation of moduli spaces of sheaves on an abelian surface [8].
- *(O'Grady's 10-dimensional example)* A 22-parameter deformation of moduli spaces of sheaves on a $K3$ surface [7].

For every irreducible symplectic manifold X, it is possible to endow $H^2(X, \mathbb{Z})$ with a lattice structure *(the Beauville-Bogomolov lattice)*. The Beauville-Bogomolov lattices of the known examples of irreducible symplectic manifolds are as follows:

- Deformation $K3^{[n]}$-type: $3U \oplus 2E_8(-1) \oplus \langle -2(n-1) \rangle$
- Generalised Kummer varieties: $3U \oplus \langle -2(n+1) \rangle$
- O'Grady's 6-dimensional example: $3U \oplus \langle -2 \rangle \oplus \langle -2 \rangle$
- O'Grady's 10-dimensional example: $3U \oplus 2E_8(-1) \oplus A_2(-1)$

2 Moduli of polarised irreducible symplectic manifolds

It is natural to consider the moduli of polarised irreducible symplectic manifolds. A polarisation on an irreducible symplectic manifold X is a choice of ample line bundle on X. An ample line bundle $\mathcal{L}$ on X can be identified with its first Chern class $c_1(\mathcal{L}) \in L$ and then with some $h \in L$. We will assume that the polarisation is primitive (i.e. h is primitive in L). In order to construct a moduli space, we fix some discrete data: the dimension $2n$, a choice of Beauville-Bogomolov lattice L and an orbit of primitive h in the orthogonal group $O(L)$ (a polarisation type). These choices define the numerical type N of the irreducible symplectic manifold. One can use the results of Viehweg to construct a moduli space $\mathcal{M}_{n,N,h}$ that parametrises the polarised irreducible symplectic manifolds $(X, \mathcal{L})$ with dimension $2n$ and Beauville-Bogomolov lattice L. The moduli space is quasi-projective and exists as a GIT quotient.

2.1 Moduli and quotients of hermitian symmetric domains of type IV by an arithmetic group

One can relate the moduli space $\mathcal{M}_{n,N,h}$ to an *orthogonal modular variety.* Given a Beauville-Bogomolov lattice L, one can define the period domain Ω_L where $\Omega_L = \{[x] \in \mathbb{P}(L) | (x,x) = 0,\ (x, \overline{x}) > 0\}$ (which is a hermitian symmetric domain of type IV) and an orthogonal group $O(L)$. The period domain Ω_L has two connected components, we pick one and call it $\mathcal{D}_L$ and and let $O^+(L)$ be its stabilser. Given a primitive $h \in L$, define $O^+(L,h)$ to be the stabiliser of h in $O^+(L)$. If $L_h = (h)^{\perp}_L$, then one can consider $O^+(L,h) < O^+(L_h)$ and form the quotient $O^+(L,h)\backslash\mathcal{D}_{L_h}$. We refer to such varieties as orthogonal modular varieties.

We have the following theorem:

Theorem 2.1 *[3] For every component $\mathcal{M}'_{n,N,h}$ of $\mathcal{M}_{n,N,h}$, there exists a finite-to-one dominant morphism from $\mathcal{M}_{n,N,h}$ to $O^+(L,h)\backslash\mathcal{D}_{L_h}$.*

This is sufficient to prove results about the birational geometry of the moduli space: in particular, one can determine the Kodaira dimension.

An orthogonal modular variety $\mathcal{F}_L$ (i.e. the quotient of a connected component of some period domain Ω_L by an arithmetic subgroup Γ of an orthogonal group $O(L)$ where L is a lattice of signature $(2,n)$) is quasi-projective (by the results of Baily and Borel) and it is also a locally symmetric variety. For our applications, it is necessary to work with smooth (or 'nearly smooth') models. One may prove existence of such models by applying Mumford's theory of toroidal compactifications [1] and, in doing so, one obtains theorems of the following kind:

Theorem 2.2 *If $n \geq 7$ then $\mathcal{F}_L$ has canonical singularities.*

3 Modular forms and Kodaira dimension

Definition 3.1 *If Y is a connected smooth projective variety of dimension n, The* Kodaira dimension *$\kappa(Y)$ of Y is defined by*

$$\kappa(Y) = tr.deg(\bigoplus_{k \geq 0} H^0(Y, kK_Y)) - 1$$

$\kappa(Y)$ can take values in $-\infty, 0, 1, \ldots, n = dim(Y)$ and Y is said to be of general type *if $\kappa(Y) = dim(Y)$. The Kodaira dimension is a bimeromorphic invariant and, for an arbitrary quasi-projective variety Z, one defines the Kodaira dimension of Z as the Kodaira dimension of a desingularisation of a compactification of Z.*

Definition 3.2 *Let L be a lattice of signature $(2,n)$ with $n \geq 3$. Let $k \in \mathbb{Z}$ and let $\chi : \Gamma \to \mathbb{C}*$ be a character for a subgroup $\Gamma < O^+(L)$ of finite index. A holomorphic function from the affine cone $\mathcal{D}^\bullet_L$ of $\mathcal{D}_L$ to $\mathbb{C}$ is called a modular form of weight k for the group Γ with character χ if*

$$F(tZ) = t^{-k}F(Z) \quad \forall t \in \mathbb{C}^*$$

$$F(gZ) = \chi(g)F(Z) \quad \forall g \in \Gamma$$

A modular form is called a cusp form if it vanishes at the cusps. We denote the linear spaces of modular forms and cusp forms of weight k for the group Γ and character χ by $M_k(\Gamma, \chi)$ and $S_k(\Gamma, \chi)$, respectively.

3.1 The low weight cusp form trick

Theorem 3.1 *[4] Let L be an integral lattice of signature $(2,n)$, $n \geq 9$. The modular variety $\mathcal{F}_L(\Gamma)$ is of general type if there exists a non-zero cusp form $F_a \in S_a(\Gamma, \chi)$ of small weight $a \geq n$ vanishing with order at least 1 at infinity such that $divF_a \geq Bdiv(\pi_\Gamma)$ where $Bdiv(\pi_\Gamma)$ denotes the branch divisor of the projection $\pi_\Gamma : \mathcal{D}_L \to \mathcal{F}_L(\Gamma)$.*

The main ideas of the argument are as follows:

If there exists a special modular form F_a, one may construct an infinite series of Γ-invariant differential forms which define sections of the pluricanonical bundle of $\mathcal{F}_L$ away from the ramification divisor of the projection from $\mathcal{D}_L$ onto $\overline{\mathcal{F}_L(\Gamma)}$ and also away from the cusps. If it is possible to exhibit such an F_a and if it is also possible to pick a compactification $\overline{\mathcal{F}_L(\Gamma)}$ of $\mathcal{F}_L(\Gamma)$ with sufficiently amenable geometric behaviour, one can extend the differential forms to a global section of the pluricanonical bundle on $\mathcal{F}_L(\Gamma)$ and apply Hirzebruch-Mumford proportionality in order to obtain the general type statement.

As a consequence of this theorem, one can prove the following:

Theorem 3.2 *[4] The moduli space $\mathcal{F}_{2d}$ of K3 surfaces with a polarisation of degree $2d$ is of general type for any $d \geq 61$ and for $d = 46, 50, 52, 54, 57, 58$ and 60. If $d \geq 40$ and $d \neq 41, 44, 45$ or 47 then the Kodaira dimension of $\mathcal{F}_{2d}$ is non-negative.*

Theorem 3.3 *[6] Let d be a positive integer not equal to 2^n with $n \geq 0$. Then every component of the moduli space of 10-dimensional O'Grady varieties with split polarisation h of Beauville degree $h^2 = 2d \neq 2^{n+1}$ is of general type.*

The cusp forms for the K3 surface case come from pullbacks of the Borcherds form Φ_{12}. The following theorems establish that the form has the properties required:

Theorem 3.4 *[4] Let $L \hookrightarrow II_{2,26}$ be a non-degenerate sublattice of signature $(2,n)$ with $n \geq 3$ where $II_{2,26}$ is the unique even unimodular lattice of signature $(2,26)$ and let $\mathcal{D}_L \hookrightarrow \mathcal{D}_{II_{2,26}}$ be the corresponding embedding of the homogeneous domains. Let Φ_{12} be the Kac-Weyl-Borcherds modular form associated with the orthogonal group $O^+(II_{2,26})$ and determinant character. Then the set of (-2)-roots*

$$R_{-2}(L^\perp) = \{r \in II_{2,26} \mid r^2 = -2,\ (r, L) = 0\}$$

in the orthogonal complement is finite. Then the function

$$\Phi|_L = \frac{\Phi_{12}(Z)}{\prod_{r \in R_{-2}(L^\perp)/\pm 1}(Z, r)}|_{\mathcal{D}_L}$$

lies in $M_{12+N(L^\perp}(\tilde{O}(L), det)$ (where, in the product over r, we fix a finite set of representatives in $R_{-2}(L^\perp)/\pm 1$). The modular form $\Phi|_L$ vanishes only on rational quadratic divisors of type $\mathcal{D}_v(L)$ where $v \in L^\vee$ is the orthogonal projection of a (-2)-root $r \in II_{2,26}$ on $L^\vee$ where $L^\vee$ is the dual lattice of L.

If the set of roots is non-empty, we refer to the modular form $\Phi|_L$ as the quasi-pullback of Φ_{12}.

Theorem 3.5 *[2] Let $L \hookrightarrow II_{2,26}$ be a non-degenerate sublattice of signature $(2,n)$. If the set of roots $R_{-2}(L^\perp)$ is finite then the quasi-pullback $\Phi|_L$ of the Borcherds form is a cusp form Φ_{12}.*

One may show that these conditions are satisfied for the K3 lattice and that the cusp form satisfies the other necessary conditions in the statement of the low weight cusp form trick.

The low weight cusp form trick uses forms of low weight ($k < n$) and large divisor ($divF \geq Bdiv(\pi_\Gamma)$) but forms of high weight ($k \geq n$) and small divisor ($divF \geq Bdiv(\pi_\Gamma)$) are also useful.

Theorem 3.6 *[5] Suppose L is a lattice of signature $(2,n)$, with $n \geq 3$. Let $F_k \in M_k(\Gamma, \chi)$ be a strongly reflective modular form of weight k and character χ for a subgroup $\Gamma < O^+(L)$ of finite index. Then,*

$$\kappa(\mathcal{F}_L(\Gamma)) = -\infty$$

if $k > n$ or if $k = n$ and F_k is not a cusp form. If k and F_n is a cusp form whose order of vanishing at infinity is at least 1 then

$$\kappa(\Gamma_\chi \backslash \mathcal{D}_L) = 0\ ,$$

where $\Gamma_\chi = ker(\chi.det)$ is a subgroup of Γ.

As a consequence of this theorem, one can find examples of lattice polarised K3 surfaces with $\kappa = -\infty$ or $\kappa = 0$.

4 My interests

- Understanding the moduli of the generalised Kummer varieties
- General interest in modular forms for $O(2,n)$ and, in particular, the existence of Borcherds-type products.

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