Hilbert functions of 0-dimensional Schemes in $\mathbb{P}^1 \times \mathbb{P}^1$

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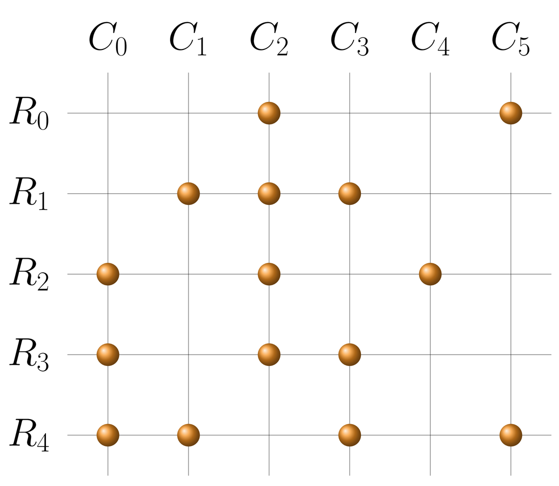
CONTRIBUTION

Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and let $X \subset Q$ be a 0-dimensional scheme. Only if X is ACM, then its Hilbert function can be easily computed. We introduce:

- an iterative method to compute the Hilbert functions of some non ACM subschemes in Q
- a class of Hilbert functions of subschemes together with their geometrical description.

INTRODUCTION

Given a reduced 0-dimensional scheme X, we call $\mathbf{R_0, \dots, R_a}$ and $\mathbf{C_0, \dots, C_b}$ the $(1,0)$ and $(0,1)$-lines containing X. Any two $(1,0)$-lines (such as any two $(0,1)$-lines) have empty intersection. Every point is the intersection of a $(1,0)$ and a $(0,1)$-line. So, usually X is represented on a grid of lines:



Given a 0-dimensional scheme $X \subset Q$, let us consider:

- $S(X) = k[\mathbb{P}^1 \times \mathbb{P}^1]/I(X)$ and $m_{ij} = \dim_k S(X)_{(i,j)}$
- $M_X = (m_{ij})$, its bi-graded Hilbert function
- $c_{ij} = m_{ij} + m_{i-1j-1} - m_{ij-1} - m_{i-1j}$ and $\Delta M_X = (c_{ij})$
- $a_{ij} = m_{ij} - m_{ij-1}$ and $b_{ij} = m_{ij} - m_{i-1j}$.

Note that $c_{ij} \le 1$ for any (i,j) and $c_{ij} = 1 \iff I(X)_{(i,j)} = 0$. Moreover, $1 \le \operatorname{depth} S(X) \le 2$. X is called arithmetically Cohen-Macaulay (**ACM** for short) if $\operatorname{depth} S(X) = 2$.

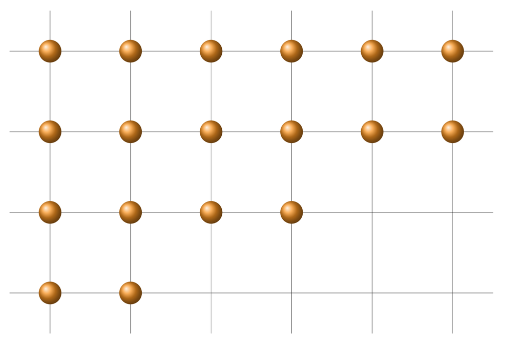
ACM CASE

CHARACTERIZATION THEOREMS

- A 0-dimensional scheme $X \subset Q$ is ACM $\iff c_{ij} \ge 0$ for any (i,j).
- If $M = (m_{ij})$ is a table such that:
  1. $c_{ij} \le 1$ and $c_{ij} = 0$ for either $i \gg 0$ or $j \gg 0$
  2. $c_{ij} \le 0 \Rightarrow c_{rs} \le 0$ for any $(r,s) \ge (i,j)$
  3. $0 \le a_{ij} \le a_{ij-1}$ and $0 \le b_{ij} \le b_{i-1j}$ for any (i,j)

  and, moreover, if $c_{ij} \ge 0$ for any (i,j), then M is the Hilbert function of an ACM scheme.

Let $X \subset Q$ be an ACM scheme. Then ΔM_X and the geometry of X are strictly related:



X

	0	1	2	3	4	5	…
0	1	1	1	1	1	1	0
1	1	1	1	1	1	1	0
2	1	1	1	1	0	0	0
3	1	1	0	0	0	0	0
⋮	0	0	0	0	0	0	0

$\Delta M_X = (c_{ij})$

CONTACTS

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COMPUTING THE HILBERT FUNCTION

THEOREM

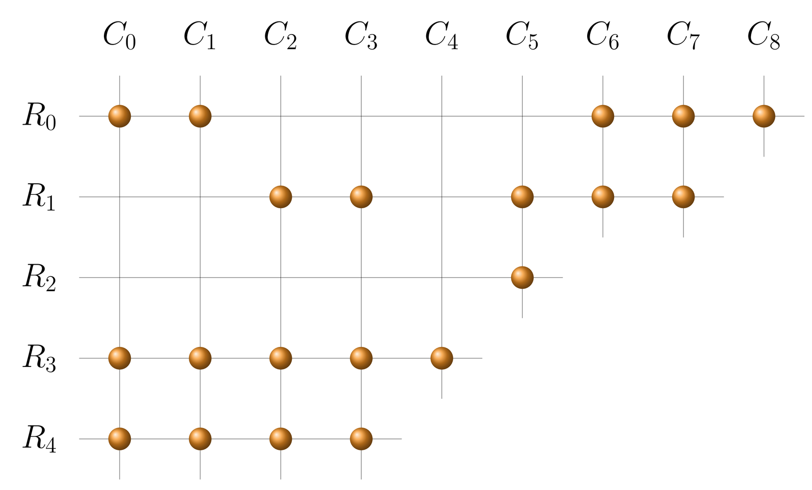
Let X be a 0-dimensional scheme and let R be a $(1,0)$-line disjoint from X. Let $C_{b+1}, \dots, C_n$, $n \ge b$, be arbitrary $(0,1)$-lines and $i_1, \dots, i_r \in \{0, \dots, b\}$. Let $\mathcal{P} = \{R \cap C_i \mid i \in \{0, \dots, n\},\ i \ne i_1, \dots, i_r\}$ and let $Z = X \cup \mathcal{P}$. Suppose also that on the $(0,1)$-line C_{i_k} there are q_k points of X for $k = 1, \dots, r$ and that $q_1 \le q_2 \le \dots \le q_r$. Then, given $T = \{(q_1, n), (q_2, n-1), \dots, (q_r, n-r+1)\}$, we have:

$$c_{ij}(Z) = \begin{cases} 1 & \text{if } i = 0,\ j \le m + r \\ 0 & \text{if } i = 0,\ j \ge m + r + 1 \\ c_{i-1j}(X) & \text{if } i \ge 1 \text{ and } (i,j) \notin T \\ c_{i-1j}(X) - 1 & \text{if } i \ge 1 \text{ and } (i,j) \in T \end{cases}$$

if one of the following conditions holds:

1. X is ACM;
2. $r = 0, 1$;
3. $r \ge 2$, $q_{r-1} < q_r$ and $\Delta M_X^{(i, m+r-k+1)} = 0$ for any $k \in \{1, \dots, r-1\}$ and $i \ge q_k$;
4. $r \ge 2$, $q_{r-1} = q_r$ and $\Delta M_X^{(i, m+r-k+1)} = 0$ for any $k \in \{1, \dots, r\}$ and $i \ge q_k$.

Thanks to this theorem it is possible to compute, recursively, the Hilbert functions of some non ACM schemes. For example:



X

	0	1	2	3	4	5	6	7	8	…
0	1	1	1	1	1	1	1	1	1	0
1	1	1	1	1	1	1	1	1	−1	0
2	1	1	1	1	1	1	0	−2	0	0
3	1	1	1	1	1	−3	−2	0	0	0
4	1	0	0	0	−1	0	0	0	0	0
⋮	0	0	0	0	0	0	0	0	0	0

$\Delta M_X = (c_{ij})$

CONDITIONS ON THE HILBERT FUNCTION

Let $M = (m_{ij})$ be a table such that:

1. $c_{ij} \le 1$ and $c_{ij} = 0$ for either $i \gg 0$ or $j \gg 0$
2. $c_{ij} \le 0 \Rightarrow c_{rs} \le 0$ for any $(r,s) \ge (i,j)$
3. $0 \le a_{ij} \le a_{ij-1}$ and $0 \le b_{ij} \le b_{i-1j}$ for any (i,j).

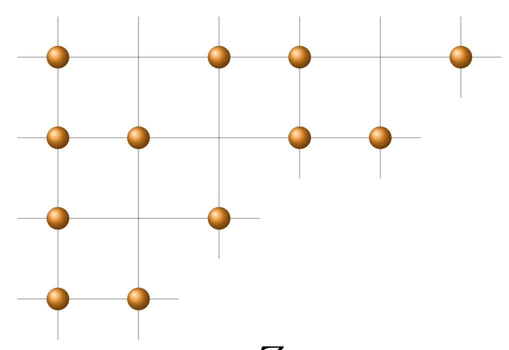
Take any (i_1, j_1) and (i_2, j_2) such that $c_{i_1 j_1}, c_{i_2 j_2} < 0$ and consider $r_1 \in \{0, \dots, -c_{i_1 j_1} - 1\}$ and $r_2 \in \{0, \dots, -c_{i_2 j_2} - 1\}$. Suppose that $i_1 \ne i_2$ and $j_1 \ne j_2$ and either $a_{i_1 j_1} + r_1 \ne a_{i_2 j_2} + r_2$ or $b_{i_1 j_1} + r_1 \ne b_{i_2 j_2} + r_2$. Then the it is possible to describe a non ACM scheme Z such that $M = M_Z$. An example is the following:

	0	1	2	3	4	5	…
0	1	2	3	4	5	6	6
1	2	4	6	8	10	10	10
2	3	6	9	11	11	11	11
3	4	8	10	12	12	12	12
⋮	4	8	10	12	12	12	12

M

	0	1	2	3	4	5	…
0	1	1	1	1	1	1	0
1	1	1	1	1	1	−1	0
2	1	1	1	0	−2	0	0
3	1	1	−1	0	0	0	0
⋮	0	0	0	0	0	0	0

ΔM



Z

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