Minimal free resolutions of 0-dimensional schemes in $\mathbb{P}^1 \times \mathbb{P}^1$

Paola Bonacini and Lucia Marino
University of Catania (Italy), Mathematics and Computer Science Department

CONTRIBUTION

Let X be a zero-dimensional scheme in $\mathbb{P}^1 \times \mathbb{P}^1$. Then X has a minimal free resolution of length 2 if and only if X is ACM. We determine a class of reduced schemes whose resolutions, similarly to the ACM case, can be obtained by their Hilbert functions and depends only on their distributions of points in a grid of lines. Moreover, a minimal set of generators of the ideal of these schemes is given by curves split into the union of lines.

INTRODUCTION

Given a 0-dimensional scheme $X \subset \mathbb{P}^1 \times \mathbb{P}^1$, let us consider:

- $S(X) = k[\mathbb{P}^1 \times \mathbb{P}^1]/I(X)$
- $m_{ij} = \dim_k S(X)_{(i,j)}$ and $M_X = (m_{ij})$, its bi-graded Hilbert function
- $c_{ij} = m_{ij} + m_{i-1j-1} - m_{ij-1} - m_{i-1j}$ and $\Delta M_X = (c_{ij})$.

It is known that $1 \le \operatorname{depth} S(X) \le 2$, so that $S(X)$ has a minimal free resolution of length ≤ 3. $S(X)$ has a minimal free resolution of length 2 when X is ACM. So we see that a minimal free resolution of a zero-dimensional scheme in $Q = \mathbb{P}^1 \times \mathbb{P}^1$ is of the following type:

$$0 \to \bigoplus_{i=1}^{t} \mathcal{O}_Q(-a_{3i}, -a'_{3i}) \to \bigoplus_{i=1}^{n} \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \to \bigoplus_{i=1}^{m} \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \to \mathcal{I}_X \to 0$$

and X is ACM if $t = 0$.

ACM CASE

It is known that for any 0-dimensional scheme $X \subset Q$ $c_{ij} \le 1$ for any i, j and a X is ACM $\iff c_{ij} \ge 0$ for any (i, j). Moreover, if $c_{ij} = 1$, then $c_{hk} = 1$ for any $h \le i$ and $k \le j$. The pair (i, j) is called:

- **corner for X** if $c_{ij} \le 0$ and $c_{ij-1} = c_{i-1j} = 1$
- **vertex for X** if $c_{ij} \le 0$, $c_{ij-1}, c_{i-1j} \le 0$ and $c_{i-1j-1} = 1$.

It is known that if X is ACM, then the generators's degrees (a_{1i}, a'_{1i}) run over all the corners of X and the first syzygies (a_{2i}, a'_{2i}) run over all the vertices of X and a set of minimal generators of X is given by curves split in the union of lines.

NOTATION

Let $X \subset Q$ be a reduced ACM 0-dimensional scheme. We call $R_0, \dots, R_a$ and $C_0, \dots, C_b$, respectively, the $(1,0)$ and $(0,1)$-lines containing X and each at least one point of X. We consider $P_{ij} = R_i \cap C_j$ for any i, j.

THEOREM

Let X be a reduced ACM 0-dimensional scheme and $P_{i_1 j_1}, \dots, P_{i_h j_h} \in X$ points such that any two if them are not contained in a $(1,0)$ or $(0,1)$-line and $X \setminus \{P_{i_r j_r}\}$ is not ACM for any $r = 1, \dots, h$. Let us consider:

$$Z = X \setminus \{P_{i_1 j_1}, \dots, P_{i_h j_h}\}.$$

Let $q_l + 1 = \#(X \cap C_l)$ and $p_l + 1 = \#(X \cap R_l)$ for $l = 1, \dots, h$ and let:

$$r_{ij} = \#\{l \in \{1, \dots, h\} \mid (q_l, p_l) = (i, j)\},$$

for any (i, j). Then the minimal free resolution of Z is:

$$0 \to \bigoplus_{l=1}^{h} \mathcal{O}_Q(-q_l - 1, -p_l - 1) \to$$
$$\to \bigoplus_{i=1}^{m-1} \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \oplus \bigoplus_{l=1}^{h} \mathcal{O}_Q(-q_l, -p_l - 1) \oplus \bigoplus_{l=1}^{h} \mathcal{O}_Q(-q_l - 1, -p_l) \to$$
$$\to \bigoplus_{i=1}^{m} \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \oplus \bigoplus_{l=1}^{h} \mathcal{O}_Q(-q_l, -p_l) \to \mathcal{I}_Z \to 0,$$

there exists a minimal set of generators of I_Z given by curves split into the union of $(1,0)$ and $(0,1)$-lines and

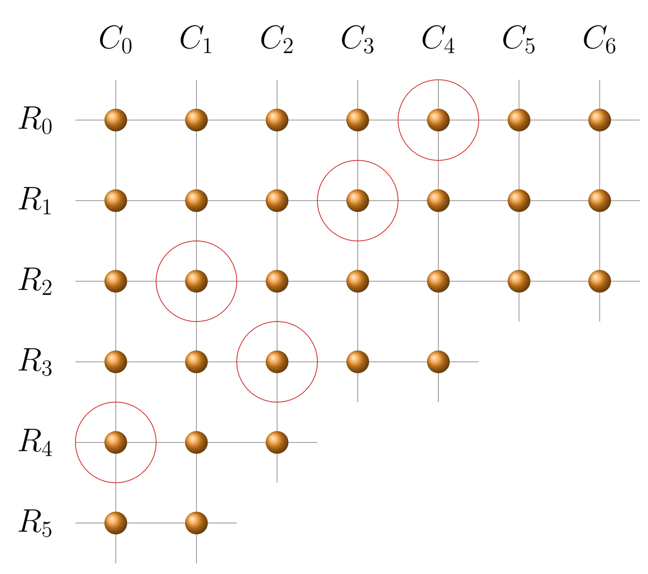
$$\Delta M_Z(i,j) = \Delta M_X(i,j) - r_{ij}$$

for any (i, j). In particular, we see that a pair (i, j) is the degree of:

1. a minimal generator for Z if and only if either (i, j) is a corner for ΔM_Z or $c_{ij} < 0$
2. a first syzygy for Z if and only if either (i, j) is a vertex for ΔM_Z or $c_{ij-1} < 0$ or $c_{i-1j} < 0$
3. a second syzygy if and only if $c_{i-1j-1} < 0$.

EXAMPLE OF A NON ACM SCHEME

Let X be a reduced ACM scheme in Q with the following configuration of points in a grid of lines and let $P_{04}, P_{13}, P_{21}, P_{32}, P_{40} \in X$:



If $Z = X \setminus \{P_{04}, P_{13}, P_{21}, P_{32}, P_{40}\}$, by the previous theorem ΔM_Z is:

	0	1	2	3	4	5	6	7	···
0	1	1	1	1	1	1	1	0	···
1	1	1	1	1	1	1	1	0	···
2	1	1	1	1	1	1	1	0	···
3	1	1	1	1	1	0	−2	0	···
4	1	1	1	0	−1	0	0	0	···
5	1	1	−1	0	0	0	−1	0	···
6	0	0	0	0	0	0	0	0	···
⋮	···	···	···	···	···	···	···	···	···

and its minimal free resolution is:

$$0 \to \mathcal{O}_Q(-6,-3) \oplus \mathcal{O}_Q(-6,-7) \oplus \mathcal{O}_Q(-5,-5) \oplus \mathcal{O}_Q(-4,-7)^{\oplus 2} \to$$
$$\to \mathcal{O}_Q(-6,-2)^{\oplus 2} \oplus \mathcal{O}_Q(-5,-3)^{\oplus 2} \oplus \mathcal{O}_Q(-4,-5)^{\oplus 2} \oplus \mathcal{O}_Q(-3,-7)^{\oplus 3} \oplus \mathcal{O}_Q(-5,-7) \oplus \mathcal{O}_Q(-6,-6) \oplus \mathcal{O}_Q(-5,-4) \oplus \oplus \mathcal{O}_Q(-4,-6)^{\oplus 2} \to$$
$$\to \mathcal{O}_Q(-6,0) \oplus \mathcal{O}_Q(-5,-2)^{\oplus 2} \oplus \mathcal{O}_Q(-4,-3) \oplus \mathcal{O}_Q(-3,-5) \oplus \mathcal{O}_Q(0,-7) \oplus \mathcal{O}_Q(-5,-6) \oplus \mathcal{O}_Q(-4,-4) \oplus \mathcal{O}_Q(-3,-6)^{\oplus 2} \to \mathcal{I}_Z \to 0.$$

REFERENCES

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CONTACTS

http://www.dmi.unict.it/~bonacini
http://www.dmi.unict.it/~lmarino
bonacini@dmi.unict.it, lmarino@dmi.unict.it