The Hodge Conjecture for products of complex algebraic surfaces

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The Hodge Conjecture

The **Hodge conjecture** was posed as one of the seven Millennium problems in 2000 and still remains uncertain. The objects of interest are smooth complex projective varieties X of some (complex) dimension n. Then $Z^k(X)$ is the free abelian group generated by irreducible subvarieties of codimension k in X. If Y is such a subvariety (maybe singular), the *fundamental class* $[Y] \in H^{2k}(X)$ is characterized by the property $\int_X \alpha \cup [Y] = \int_Y \alpha|_Y$, for $\alpha \in H^{2n-2k}(X)$. This defines the *cycle class map*

$$\Psi : Z^k(X) \to H^{2k}(X, \mathbb{Z}), \quad Z = \sum a_i Z_i \mapsto [Z] := \sum a_i [Z_i].$$

Hodge decomposition yields that any cohomology group $H^i(X, \mathbb{C})$ decomposes into a direct sum of sheaf cohomology groups, i.e.

$$H^i(X, \mathbb{C}) \cong \bigoplus_{p+q=i} H^{p,q}(X)$$

It is easy to see that the fundamental class $[Z]$ is contained in $H^{k,k}(X, \mathbb{Z}) := H^{k,k}(X) \cap H^{2k}(X, \mathbb{Z})$. So the cycle class map factors over $H^{k,k}(X, \mathbb{Z})$. Classes which lie in the middle part of the decomposition and which are rational, i.e. classes in $H^{k,k}(X, \mathbb{Q}) := H^{k,k}(X) \cap H^{2k}(X, \mathbb{Q})$ are called *Hodge classes*. By the above any fundamental class of a cycle is a Hodge class and Hodge classes are called *algebraic* if they are rational linear combinations of classes of algebraic cycles.

The \$1,000,000 Question. (Hodge Conjecture) *Is every Hodge class algebraic? Or in other words, is* $\Psi_{\mathbb{Q}} : Z^k(X)_{\mathbb{Q}} \to H^{k,k}(X, \mathbb{Q})$ *surjective?*

Naturally one would also like to understand the kernel of the cycle class map. This concerns another conjecture, namely the Bloch-Beilinson conjecture. But let us see what is already known about the image.
For integral X its fundamental class generates the cohomology group $H^0(X, \mathbb{Z})$. As our varieties are always assumed to be integral all $(0,0)$-Hodge classes are then rational multiples of $[X]$, i.e. algebraic. The same works for the top cohomology group $H^{2n}(X, \mathbb{Z})$ which is generated by the class of any point, so that any (n,n)-Hodge class is algebraic as well. Considering $(1,1)$-Hodge classes Lefschetz' theorem yields that the integral Hodge classes are given by integral linear combinations of algebraic classes.

Lefschetz theorem on $(1,1)$-classes *The following map is surjective*

$$Z^1(X)/\sim_{\text{lin}} \cong \text{Pic}(X) \xrightarrow{c_1} H^{1,1}(X, \mathbb{Z}), \ D \mapsto \mathcal{O}_X(D) \overset{c_1}{\mapsto} [D].$$

While this is true for integral $(1,1)$-classes, there are counter examples for the integral Hodge conjecture in general. Atiyah and Hirzebruch [AH62] showed the existence of a torsion Hodge class which is non-algebraic, and Kollár [Kol92] found a non-torsion integral Hodge class which is not algebraic, but which has an integral multiple which is algebraic. Another interesting aspect comes from the hard Lefschetz theorem

Hard Lefschetz theorem *Let $k \leq n$ and ω rational Kähler class. Then $L^{n-k} : H^k(X, \mathbb{Q}) \to H^{2n-2k}(X, \mathbb{Q})$, induced by $\alpha \mapsto \alpha \wedge \omega^{n-k}$ on forms, is an isomorphism respecting bidegrees.*

From this we can deduce, that the $(n-k, n-k)$-Hodge classes are algebraic if the (k,k)-Hodge classes are algebraic and $2k \leq n$.

Self products of K3 surfaces

The above statements hence yield the Hodge conjecture for varieties of dimension up to $n = 3$. The first interesting case occurs for $(2,2)$-classes in dimension 4.

n	(0,0)	(1,1)	(2,2)	(3,3)	(4,4)
0	✓	-	-	-	-
1	✓	✓	-	-	-
2	✓	✓	✓	-	-
3	✓	✓	✓	✓	-
4	✓	✓	?	✓	✓

So we know that Hodge classes on a surface are algebraic. Does this statement then also hold for products $X \times Y$ of surfaces?

Introduce the birational invariants $q := \dim H^{0,1}(X)$ (*irregularity*) and $p_g := \dim H^{0,2}(X)$ (*geometric genus*).

Definition. A smooth complex projective surface S is a *K3 surface* if its canonical bundle $\omega_S \cong \mathcal{O}_S$ is trivial and $H^1(S, \mathcal{O}_S) \cong H^{0,1}(S) = 0$. In particular a K3 surface has invariants $q = 0$ and $p_g = 1$.

We consider the self-product of a K3 surface $S \times S$. Künneth decomposition and the vanishing of the respective cohomology groups yield

$$\begin{aligned} H^{2,2}(S \times S, \mathbb{Q}) &\cong H^0(S, \mathbb{Q}) \otimes H^4(S, \mathbb{Q}) \oplus H^4(S, \mathbb{Q}) \otimes H^0(S, \mathbb{Q}) \\ &\oplus \left(\text{End}_{\mathbb{C}}(H^{2,0}(S)) \oplus \text{End}_{\mathbb{C}}(H^{1,1}(S)) \oplus \text{End}_{\mathbb{C}}(H^{0,2}(S))\right) \cap \text{End}_{\mathbb{Q}}(H^2(S, \mathbb{Q})) \\ &\cong \text{algebraic cycles} \oplus \text{End}_{\text{Hdg}}(H^2(S, \mathbb{Q})) \end{aligned}$$

where an endomorphism of the Hodge structure $H^2(S, \mathbb{Q})$ is exactly an endomorphism of the $\mathbb{Q}$-vector space $H^2(S, \mathbb{Q})$ which respects the bidegree decomposition on $H^2(S, \mathbb{C})$.
For a smooth complex projective surface X let $\text{Pic}(X)$ be its Picard group and let $\rho(X)$ denote its rank. The image of $\text{Pic}(X)$ in $H^2(X, \mathbb{Z})$ is called the *Néron–Severi group* $\text{NS}(X)$. From the Lefschetz (1,1)-theorem we get $\text{NS}(X)_{\mathbb{Q}} = H^{1,1}(X, \mathbb{Q})$. Via the intersection pairing we define the *transcendental lattice* as $T(X) := \text{NS}(X)^{\perp} \subset H^2(X, \mathbb{Z})$. We get an orthogonal decomposition

$$H^2(X, \mathbb{Q}) = \text{NS}(X)_{\mathbb{Q}} \oplus T(X)_{\mathbb{Q}}.$$

Due to the fact that $T(S)_{\mathbb{Q}}$ does not contain any non-trivial proper sub Hodge structure we get $\text{Hom}_{\text{Hdg}}(NS(S)_{\mathbb{Q}}, T(S)_{\mathbb{Q}}) = 0$, $\text{Hom}_{\text{Hdg}}(T(S)_{\mathbb{Q}}, \text{NS}(S)_{\mathbb{Q}}) = 0$ and so

$$\text{End}_{\text{Hdg}}(H^2(S, \mathbb{Q})) \cong \text{End}_{\text{Hdg}}(\text{NS}(S)_{\mathbb{Q}}) \oplus \text{End}_{\text{Hdg}}(T(S)_{\mathbb{Q}}) \cong \text{ algebraic cycles} \oplus \text{End}_{\text{Hdg}}(T(S)_{\mathbb{Q}}).$$

Question. (Hodge Conjecture for $S \times S$) *Is every class in* $\text{End}_{\text{Hdg}}(T(S)_{\mathbb{Q}})$ *algebraic?*

By the same reasoning any non-trivial endomorphism $T(S)_{\mathbb{Q}} \to T(S)_{\mathbb{Q}}$ must be an isomorphism. In the question which of these are algebraic we could first restrict ourselves to the easier case of *Hodge isometries*, i.e. isomorphisms of Hodge structures which respect the intersection product. Let S' be another K3 surface. Then Mukai has proven the following theorems in [5].

Theorem 2. *For $\rho(S) \geq 11$ every Hodge isometry $T(S)_{\mathbb{Q}} \to T(S')_{\mathbb{Q}}$ is algebraic.*

Theorem 3. *Any integral Hodge isometry $T(S) \to T(S')$ is algebraic.*

Geometric genus 1 and associated K3 surfaces

Let us turn to the general case of a product of algebraic surfaces $X \times Y$. The interesting Hodge classes are as before given by $H^{2,2}(X \times Y, \mathbb{Q})$ which is contained in

$$\text{Hdg}(X \times Y) := \left(\bigoplus H^i(X \times Y, \mathbb{Q})\right) \cap \left(\bigoplus H^{p,p}(X \times Y)\right)$$

and again by the Künneth isomorphism we get for $i + j = 4$

$$H^{2,2}(X \times Y) \cap (H^i(X, \mathbb{Q}) \otimes H^j(Y, \mathbb{Q})) \cong \text{Hom}_{\text{Hdg}}(H^i(X, \mathbb{Q})^*, H^j(Y, \mathbb{Q})(2))$$

Note that the bilinear form on $T(X)$ determines a natural isomorphism of rational Hodge structures $T(X)_{\mathbb{Q}} \cong T(X)^*_{\mathbb{Q}}(-2)$. Analogous to the case of interest before, the *Hodge–Künneth Transcendence group* of X and Y is defined by

$$\text{HKT}(X, Y) := \text{Hdg}(X \times Y) \cap (T(X)_{\mathbb{Q}} \otimes T(Y)_{\mathbb{Q}}) \cong \text{Hom}_{\text{Hdg}}(T(X)^*_{\mathbb{Q}}, T(Y)_{\mathbb{Q}}(2)) \cong \text{Hom}_{\text{Hdg}}(T(X)_{\mathbb{Q}}, T(Y)_{\mathbb{Q}}).$$

Note that in the case of surfaces with geometric genus 1 the transcendental lattice is again irreducible over $\mathbb{Q}$, so that every such homomorphism is either the zero map or an isomorphism. Due to arguments given by Lieberman and Okamoto we can again reduce to

Question. (Hodge Conjecture for $X \times Y$) *Is every class in* $\text{Hom}_{\text{Hdg}}(T(X)_{\mathbb{Q}}, T(Y)_{\mathbb{Q}})$ *algebraic?*

Being algebraic as a homomorphism $\zeta : T(X)_{\mathbb{Q}} \to T(Y)_{\mathbb{Q}}$ corresponds to the fact that there is an algebraic cycle W on $X \times Y$ such that $\zeta : \alpha \mapsto \text{pr}_{2*}(\text{pr}_1^*(\alpha) \cup W)$.
To any surface with $p_g = 1$ Morrison associated a K3 surface with the same transcendental lattice.

Theorem 4. (Morrison) *Let X be an algebraic surface with geometric genus one.*

(i) There exists an algebraic K3 surface S and an integral Hodge isometry $T(X) \cong T(S)$ between X and S. In this case S is called an associated K3 surface of X.

(ii) If the minimal model of X is neither a K3 surface nor a logarithmic transform of an elliptic K3 surface, then any two associated K3 surfaces of X are isomorphic.

The aim is now to construct algebraic integral Hodge isometries from a surface with $p_g = 1$ to its associated K3 surface. We denote by K the canonical class.

Examples for surfaces with $p_g = q = 1$.

1. with $K^2 = 6$ found by Rito [6] and given as the desingularization of a double cover of a Kummer K3 surface.
2. with $K^2 = 3, 4, 5$ (see [3]) are given by the desingularization of a bidouble cover of $\mathbb{P}^2$ branched over 3 curves D_1, D_3, D_5 of respective degrees 1, 3 and 5.
3. with $K^2 = 2$ are classified by [2] and given as double covers over the symmetric square of the Albanese.

Proposition 5. *Let X and Y be smooth complex projective surfaces of geometric genus 1 and $\pi : X \to Y$ a double cover. Then there is an algebraic Hodge isometry $T(X)(2)_{\mathbb{Q}} \cong T(Y)_{\mathbb{Q}}$.*

Work in Progress

In the above cases we claim to get the following

1. The associated K3 surface is given by the associated K3 surface of the abelian surface underlying the Kummer surface.
2. The associated K3 surface is obtained as follows: Consider the double cover of $\mathbb{P}^2$ branched over D_1 and D_5. This is a Kummer K3 surface. Then the associated K3 surface of the respective abelian surface yields the desired.
3. Work in progress...

Nodal hypersurfaces. Another interesting question concerning the Hodge conjecture is the search for nodal hypersurfaces, due to Thomas in [7]. In the special case of a self-product of K3 surfaces one can restrict the search to particular families of curves as in

Theorem 6. **[1]** *Let S be a K3 surface. The Hodge conjecture for $S \times S$ is equivalent to the following statement: For any primitive $0 \neq \zeta \in \text{End}_{\text{Hdg}}(T(S))$ there is a nodal hypersurface $D \subset S \times S$, in $|\mathcal{O}(N)|$ for some N, such that $\zeta|_D \neq 0$ and $\text{pr}_i|_D : D \to S$ is a flat family of curves for $i = 1, 2$.*

Recall that a cohomology class of degree k is called *primitive* if it is in the kernel of L^{n-k+1}. The question now is to find families of curves for a given endomorphism of the transcendental lattice.

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