Derived autoequivalences of hyperkähler varieties

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Summary

Introduction

Any good story about derived categories tends to start with a result by Bondal & Orlov: In the late 90's, they showed that smooth projective varieties X which lie at the extreme ends of the curvature spectrum, i.e. ample canonical or anti-canonical bundle, have rather simple derived categories $\mathcal{D}(X) := \mathcal{D}^b(\mathrm{Coh}(X))$.

More precisely, they proved that the group of autoequivalences (which essentially reflects the complexity of $\mathcal{D}(X)$) is trivial:

$$\mathrm{Aut}(\mathcal{D}(X)) \simeq \mathrm{Aut}(X) \ltimes (\mathrm{Pic}(X) \oplus \mathbb{Z})$$

and the only Fourier-Mukai partner of such a variety is itself.

This result naturally leads us to ask what the autoequivalence group is when X is flat, i.e. zero curvature or $c_1(X) = 0$.

Introduction

Theorem (Beauville-Bogomolov decomposition)

Let X be a smooth projective variety with $c_1(X) = 0$. Then there exists a finite étale cover $\widetilde{X} \to X$ such that

$$\widetilde{X} \simeq \prod_i A_i \times \prod_j Y_j \times \prod_k Z_k$$

where the

- *A_i are simple abelian varieties,*
- *Y_j are hyperkähler (=irreducible holomorphic symplectic),*
- *Z_k are (strict) Calabi-Yau varieties of dimension at least three,*

$\Rightarrow A, Y, Z$ are the building blocks of all varieties with $c_1(X) = 0$.

Question: Can we describe $\mathrm{Aut}(\mathcal{D}(A))$, $\mathrm{Aut}(\mathcal{D}(Y))$, $\mathrm{Aut}(\mathcal{D}(Z))$?

Introduction

Theorem (Orlov)

Let A be an abelian variety and $\hat{A} := \mathrm{Pic}^0(A)$ be its dual variety. If $\rho : \mathrm{Aut}(\mathcal{D}(A)) \to \mathrm{GL}(H^*(A,\mathbb{Z})); \Phi \mapsto \Phi^H$ then we have a short exact sequence

$$0 \to 2\mathbb{Z} \oplus (A \times \hat{A}) \to \mathrm{Aut}(\mathcal{D}(A)) \to \mathrm{Im}(\rho) \to 1$$

where $2\mathbb{Z} \oplus (A \times \hat{A})$ is generated by shifts, translations and twists by line bundles $\mathcal{L} \in \mathrm{Pic}^0(A)$.

However, for a K3 surface S, which is the simplest hyperkähler variety (and a strict Calabi-Yau variety of dimension two), the problem is much more subtle.

Introduction

Conjectural answer by Bridgeland in terms of stability conditions. The difficulty lies in describing those autoequivalences which act trivially on cohomology, i.e.

$$\mathrm{Aut}^0(\mathcal{D}(S)) := \ker\left(\mathrm{Aut}(\mathcal{D}(S)) \to \mathrm{Aut}(H^*(S,\mathbb{Z}))\right) = \ \textbf{?}$$

This group is non-trivial since it contains the double shift [2] but more interestingly, if $\mathcal{E} \in \mathcal{D}(S)$ is a spherical object, i.e. $\mathrm{Ext}^*(\mathcal{E},\mathcal{E}) \simeq H^*(S^2,\mathbb{C}) \simeq \mathbb{C} \oplus \mathbb{C}[-2]$, the Seidel-Thomas twist

$$T_{\mathcal{E}} := \mathrm{cone}(\mathrm{Hom}(\mathcal{E},\ \ ) \otimes \mathcal{E} \xrightarrow{\mathrm{ev}} \mathrm{id}_S) \in \mathrm{Aut}(\mathcal{D}(S))$$

gets mapped to the reflection $v \mapsto v + (v(\mathcal{E}), v)v(\mathcal{E})$ in the hyperplane orthogonal to $v(\mathcal{E}) \in \mathrm{Aut}(H^*(S,\mathbb{Z}))$. In other words, we have $(T_{\mathcal{E}}^H)^2 \simeq \mathrm{id}_{H^*(S,\mathbb{Z})}$ and hence $T_{\mathcal{E}}^2 \in \mathrm{Aut}^0(\mathcal{D}(S))$.

It is expected that $\mathrm{Aut}^0(\mathcal{D}(S))$ is generated by spherical twists.

Introduction

Today: we will focus on autoequivalences of hyperkähler varieties.

Examples

1. Hilbert schemes $S^{[n]}$ of points on a K3 surface S.
2. Generalised Kummer variety K_n assoc to an abelian surface A. Recall, K_n is defined to be the fibre of the Albanese map
$$m : A^{[n+1]} \xrightarrow{\mu} A^{(n+1)} \xrightarrow{\Sigma} A$$
over zero, i.e. $A^{[n+1]} \supset K_n := m^{-1}(0)$.
3. Two sporadic examples of dimension six and ten which are desingularisations of specific moduli spaces of sheaves.

Up to deformation, these are all the hyperkähler varieties we know!

$\mathbb{P}^n$-objects

Definition

An object $\mathcal{E} \in \mathcal{D}(X)$ is called a $\mathbb{P}^n$-object if $\mathcal{E} \otimes \omega_X \simeq \mathcal{E}$ and $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E}) \simeq \mathbb{C}[h]/h^{n+1}$ is isom as a graded ring to $H^*(\mathbb{P}^n, \mathbb{C})$.

$$\begin{array}{ccccc}
\mathcal{E}^\vee \boxtimes \mathcal{E}[-2] & \xrightarrow{\tilde{H}} & T[-1] & \longrightarrow & \mathcal{P}_{\mathcal{E}}[-1] \\
\| & & \downarrow & & \downarrow \\
\mathcal{E}^\vee \boxtimes \mathcal{E}[-2] & \xrightarrow{H := h^\vee \boxtimes \mathrm{id} - \mathrm{id} \boxtimes h} & \mathcal{E}^\vee \boxtimes \mathcal{E} & \longrightarrow & \mathrm{cone}(H) \\
 & & \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle \tilde{\mathrm{tr}}} \\
 & & \mathcal{O}_\Delta & = & \mathcal{O}_\Delta
\end{array}$$

$$\mathcal{P}_{\mathcal{E}} := \mathrm{cone}(\mathcal{E}^\vee \boxtimes \mathcal{E}[-1] \xrightarrow{\tilde{H}[1]} T)$$

$$\simeq \mathrm{cone}(\mathrm{cone}(\mathcal{E}^\vee \boxtimes \mathcal{E} \xrightarrow{H} \mathcal{E}^\vee \boxtimes \mathcal{E}) \xrightarrow{\tilde{\mathrm{tr}}} \mathcal{O}_\Delta)$$

$\mathbb{P}^n$-objects

Theorem (Huybrechts, Thomas)

$\mathcal{P}_{\mathcal{E}}$ gives rise to a non-trivial autoequivalence of $\mathcal{D}(X)$.

Idea of proof.

Observe that $\Omega := \mathcal{E} \cup \mathcal{E}^{\perp}$ is a spanning class of $\mathcal{D}(X)$ and $P_{\mathcal{E}}$ acts on $\mathcal{E}$ by $[-2n]$ and $\mathcal{E}^{\perp}$ by the identity. Now use the criterion for equivalences (although this part is vacuously satisfied for us). □

Examples

1. Any line bundle $\mathcal{L}$ on a hyperkähler manifold X is a $\mathbb{P}^n$-object. Indeed, $\mathrm{Ext}^*_X(\mathcal{L}, \mathcal{L}) \simeq H^*(X, \mathcal{O}_X) \simeq H^*(\mathbb{P}^n, \mathbb{C})$.
2. Let S be a K3 surface and $\mathbb{P}^1 \simeq C \hookrightarrow S$. Then $\mathcal{O}_C \in \mathcal{D}(S)$ is a $\mathbb{P}^1$-object. ($\mathcal{O}_C$ is also spherical and $T^2_{\mathcal{O}_C} \simeq P_{\mathcal{O}_C}$.)
3. By the same token, we have $\mathbb{P}^n \simeq C^{[n]} \hookrightarrow S^{[n]}$ is a $\mathbb{P}^n$-object.

$\mathbb{P}^n$-objects

Theorem (Ploog)

There is an injective group homomorphism

$$\mathrm{Aut}(\mathcal{D}(S)) \hookrightarrow \mathrm{Aut}(\mathcal{D}(S^{[n]})) \; ; \; \Phi_{\mathcal{P}} \mapsto \Phi^{S_n}_{\mathcal{P}^{\boxtimes n}}$$

by pulling back to the n-fold product, equipping with the natural S_n-action and applying the Bridgeland-King-Reid equivalence.

Examples

1. If $\mathcal{E} \in \mathcal{D}(S)$ is a $\mathbb{P}^1$-object then we have a HT-twist $P_{\mathcal{E}} = \Phi_{\mathcal{P}_a}$. Ploog tells us that this, in turn, gives rise to an auto $\Phi^{S_n}_{\mathcal{P}_a^{\boxtimes n}}$.
2. $\mathcal{E}^{\boxtimes n}$ is a $\mathbb{P}^n$-object and we have a HT-twist $P_{\mathcal{E}^{\boxtimes n}} = \Phi_{\mathcal{P}_b}$. This auto is not the same as $\Phi^{S_n}_{\mathcal{P}_a^{\boxtimes n}}$ since there are non-zero objects for which $\Phi^{S_n}_{\mathcal{P}_a^{\boxtimes n}}$ acts by $\mathrm{id}, [-2], \ldots, [-2n]$ whereas $\Phi_{\mathcal{P}_b}$ acts by $[-2n]$ on $\mathcal{E}$ and the identity on $\mathcal{E}^{\perp}$.

$\mathbb{P}^n$-functors

Definition

An exact functor $F : \mathcal{A} \to \mathcal{B}$ between triangulated categories with left and right adjoints $L, R : \mathcal{B} \to \mathcal{A}$ is a $\mathbb{P}^n$-functor if the following conditions are satisfied:

(i) There is an autoequivalence H of $\mathcal{A}$ such that

$$RF \simeq \mathrm{id} \oplus H \oplus H^2 \oplus \cdots \oplus H^n$$

(ii) The monad structure map $HRF \hookrightarrow RFRF \xrightarrow{R\epsilon F} RF$ models the algebra structure of $H^*(\mathbb{P}^n, \mathbb{C})$:

$$\mathbb{C} \cdot h^n \hookrightarrow \mathbb{C}[h]/h^{n+1} \xrightarrow{\cdot h} \mathbb{C}[h]/h^{n+1} \twoheadrightarrow \mathbb{C} \cdot 1.$$

(iii) $R \simeq H^n L$. If $\mathcal{A}$ and $\mathcal{B}$ have Serre functors, this condition is equivalent to $S_{\mathcal{B}} F H^n \simeq F S_{\mathcal{A}}$.

$\mathbb{P}^n$-functors

Similar to $\mathbb{P}^n$-objects but we replace H by the composition

$$f : FHR \hookrightarrow FRFR \xrightarrow{\epsilon FR - FR\epsilon} FR$$

$$\begin{array}{ccccc}
FHR & \xrightarrow{\tilde{f}} & T[-1] & \longrightarrow & P_F[-1] \\
\| & & \downarrow & & \downarrow \\
FHR & \xrightarrow{f := (\epsilon FR - FR\epsilon)\circ FjR} & FR & \longrightarrow & \mathrm{cone}(f) \\
 & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \tilde{\epsilon}} \\
 & & \mathrm{id}_{\mathcal{B}} & = & \mathrm{id}_{\mathcal{B}}
\end{array}$$

The composition $\epsilon \circ f$ is zero so we can find a lift $\tilde{f}$ and take the double cone but, unlike the $\mathbb{P}^n$-object case, this lift is not unique.

$$\begin{aligned} P_F &:= \mathrm{cone}(FHR[1] \xrightarrow{\tilde{f}[1]} T) \\ &\simeq \mathrm{cone}(\mathrm{cone}(FHR \xrightarrow{f} FR) \xrightarrow{\tilde{\epsilon}} \mathrm{id}_{\mathcal{B}}) \end{aligned}$$

$\mathbb{P}^n$-functors

Theorem (Addington)

If $\mathcal{B}$ is indecomposable then $P_F \in \mathrm{Aut}(\mathcal{B})$.

Proof.

Observe that $\Omega := \mathrm{im}\ F \cup (\mathrm{im}\ F)^\perp$ is a spanning class of $\mathcal{B}$ and P_F acts on $\mathrm{im}\ F$ by $H^{n+1}[2]$ and $\mathrm{im}\ F^\perp$ by the identity. Now use the criterion for equivalences. □

Examples

1. Let $\mathcal{E} \in \mathcal{D}(X)$ be a $\mathbb{P}^n$-object. $F := \mathcal{E} \otimes (\ \ ) : \mathcal{D}(\mathrm{pt}) \to \mathcal{D}(X)$ is a $\mathbb{P}^n$-functor. Indeed, $R \simeq \mathrm{Hom}(\mathcal{E},\ \ )$ and $RF \simeq \mathrm{Ext}^*(\mathcal{E},\mathcal{E})$ This recovers the Huybrechts-Thomas twist from before.
2. $q : E \to Z$ a $\mathbb{P}^n$-bundle and $i : E \to \Omega^1_q$ the zero section of the relative cotangent bundle. Then $F := i_* p^*$ is a $\mathbb{P}^n$-functor with $H = [-2]$. Family version of the embedded $\mathbb{P}^n$ example.

Hilbert schemes

Theorem (Addington)

Let $S^{[n]}$ be the Hilbert scheme of points on a smooth proj K3 surface S and $F : \mathcal{D}(S) \to \mathcal{D}(S^{[n]})$ be the natural functor induced by the universal sheaf on $S \times S^{[n]}$. Then

$$F : \mathcal{D}(S) \to \mathcal{D}(S^{[n]}) \textit{ is a } \mathbb{P}^{n-1}\textit{-functor with } H = [-2].$$

In particular, we have a non-trivial derived autoequivalence

$$P_F := \mathrm{cone}(\mathrm{cone}(FR[-2] \to FR) \to \mathrm{id}_{S^{[n]}}) \in \mathrm{Aut}(\mathcal{D}(S^{[n]}))$$

We see that P_F acts on $\mathrm{im}\ F$ by $[2-2n]$ and $(\mathrm{im}\ F)^{\perp}$ by the identity. Therefore, it cannot be a HT-twist or one coming from Ploog's construction and so $P_F \in \mathrm{Aut}(\mathcal{D}(S^{[n]}))$ is a new auto.

Hilbert schemes

Idea of proof.

Let $\mathcal{Z} \subset S \times S^{[n]}$ be the universal subscheme. Then the structure sequence $0 \to \mathcal{I}_{\mathcal{Z}} \to \mathcal{O}_{S \times S^{[n]}} \to \mathcal{O}_{\mathcal{Z}} \to 0$ gives rise to natural triangles of FM transforms $F \to F' \to F''$ and $R'' \to R' \to R$

$$\begin{array}{ccccc} R''F & \longrightarrow & R'F & \longrightarrow & \color{red}{RF} \\ \downarrow & & \downarrow & & \downarrow \\ R''F' & \longrightarrow & R'F' & \longrightarrow & RF' \\ \downarrow & & \downarrow & & \downarrow \\ R''F'' & \longrightarrow & R'F'' & \longrightarrow & RF'' \end{array}$$

Key: Ellingsrud & Strømme's work on the nested Hilbert scheme $S^{[n-1,n]}$ allows us to calculate $R''F''$ and then RF. Next, we can identify RF with $\pi_{1*}\pi_1^*$ where $\pi_1 : S^n \to S$. That is, the monad structure $RFRF \xrightarrow{R\epsilon F} RF$ is given by multiplication in $H^*(\mathcal{O}_{S^{[n-1]}})$.

Generalised Kummers

Theorem (M)

Let K_n be the generalised Kummer variety associated to an abelian surface A and $F_K : \mathcal{D}(A) \to \mathcal{D}(K_n)$ be the natural functor induced by the universal sheaf on $A \times K_n$. Then

$$F_K : \mathcal{D}(A) \to \mathcal{D}(K_n) \text{ is a } \mathbb{P}^{n-1}\text{-functor for all } n > 1.$$

In particular, we have a non-trivial derived autoequivalence

$$P_{F_K} := \mathrm{cone}(\mathrm{cone}(F_K R_K[-2] \to F_K R_K) \to \mathrm{id}_{K_n}) \in \mathrm{Aut}(\mathcal{D}(K_n))$$

We see that P_{F_K} acts on $\mathrm{im}\, F_K$ by $[2-2n]$ and $(\mathrm{im}\, F_K)^\perp$ by id. Therefore, it cannot be a HT-twist so $P_{F_K} \in \mathrm{Aut}(\mathcal{D}(S^{[n]}))$ is new.

Note that we do not (yet) have an analogue of Ploog's construction for the generalised Kummer, i.e. $\mathrm{Aut}(\mathcal{D}(A)) \overset{?}{\hookrightarrow} \mathrm{Aut}(\mathcal{D}(K_n))$.

Generalised Kummers

Idea of proof.

Cannot use nested generalised Kummer varieties, because the natural incidence variety has the wrong dimension. Instead, we study the natural functor $F : \mathcal{D}(A) \to \mathcal{D}(A^{[n+1]})$, understand why this is not a $\mathbb{P}^n$-functor for any $n > 0$ and then patiently track the special subvariety $K_n \subset A^{[n+1]}$ through $A^{[n,n+1]}$.

$$\begin{array}{ccc} A \times K_n & \xrightarrow{\ \nu\ } & A^{[n+1]} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle m} \\ A & \xrightarrow[\ \varphi\ ]{} & A \end{array} \qquad \begin{array}{ccc} (x, \zeta) & \overset{\nu}{\longmapsto} & t_x\zeta \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle m} \\ x & \underset{\varphi}{\longmapsto} & (n+1)x \end{array}$$

Key: The whole result hinges on understanding the Albanese map. That is, $m_*\mathcal{O}_{A^{[n+1]}} \simeq \bigoplus_{i=0}^n \mathcal{O}_A[-2i]$ is a formal object in $\mathcal{D}(A)$ and the monad structure $m_*m^*m_*m^* \to m_*m^*$ can be identified with multiplication in the graded ring $H^*(\mathcal{O}_{K_n})$. More precisely, we have □

Generalised Kummers

Theorem (M)

Let $m : A^{[n]} \to A$ be the Albanese map. Then

$$m^* : \mathcal{D}(A) \to \mathcal{D}(A^{[n]}) \text{ is a } \mathbb{P}^{n-1}\text{-functor with } H = [-2].$$

In particular, we obtain new derived autoequivalences

$$P_{m^*} := \mathrm{cone}(\mathrm{cone}(m^* m_*[-2] \to m^* m_*) \to \mathrm{id}_{A^{[n]}}) \in \mathrm{Aut}(\mathcal{D}(A^{[n]})).$$

Question: Mukai: $\mathcal{D}(A) \simeq \mathcal{D}(\hat{A})$. Ploog: $\mathcal{D}(A^{[n+1]}) \simeq \mathcal{D}(\hat{A}^{[n+1]})$. Does this derived equivalence respect the Beauville-Bogomolov decomposition? That is, are each of the factors of the finite étale covers derived equivalent, i.e. do we have $\mathcal{D}(K_n(A)) \simeq \mathcal{D}(K_n(\hat{A}))$?

More $\mathbb{P}^n$-functors

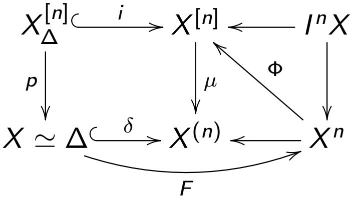
Theorem (Krug//Donovan)

Let X be any smooth proj surface and $\delta : X \to X^n$ be the diagonal embedding. Then

$$\delta_* \circ \mathrm{triv} : \mathcal{D}(X) \xrightarrow{\mathrm{triv}} \mathcal{D}^{S_n}(X) \xrightarrow{\delta_*} \mathcal{D}^{S_n}(X^n) \simeq \mathcal{D}(X^{[n]})$$

$$\textit{is a } \mathbb{P}^{n-1}\textit{-functor with } H \simeq S_X^{-1} \simeq \omega_X^{\vee}[-2]$$

In particular, we get new derived autos $P_F \in \mathrm{Aut}(\mathcal{D}(S^{[n]}))$.



When $n = 2$, this agrees with Horja's EZ-spherical twist: $\Phi \circ F \simeq i_* \circ p^*$.

Open questions

1. How much of $\mathrm{Aut}(\mathcal{D}(S^{[n]}))$ is covered by Huybrechts-Thomas, Ploog, Addington and Krug?
2. How much of $\mathrm{Aut}(\mathcal{D}(K_n))$ is covered by Huybrechts-Thomas and M? Is there an analogue of Ploog's map for the generalised Kummer?
3. Do our $\mathbb{P}^n$-functors deform? Are $F : \mathcal{D}(S) \to \mathcal{D}(\mathcal{M}_S^H(v))$ and $F : \mathcal{D}(A) \to \mathcal{D}(\mathcal{K}_A^H(v))$ $\mathbb{P}^{\frac{1}{2}\dim -1}$-functors?
4. Are there natural $\mathbb{P}^n$-functor associated to the O'Grady spaces, i.e. $\widetilde{\mathcal{M}}_S^H(2v) \rightsquigarrow \mathrm{O'G}_{10}$ and $\widetilde{\mathcal{K}}_A^H(2v) \rightsquigarrow \mathrm{O'G}_6$?
5. According to the hyperkähler SYZ conjecture, every hyperkähler manifold can be deformed into a hyperkähler manifold which admits a lagrangian fibration. Therefore, one could also investigate whether there is a natural $\mathbb{P}^n$-functor associated to a lagrangian fibration $\pi : X \to \mathbb{P}^n$.

Thanks for listening!