Using the internal language of topoi in algebraic geometry

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Abstract

There are several important topoi associated to a scheme, for instance the petit and gros Zariski topoi. These come with an internal mathematical language which closely resembles the usual formal language of mathematics, but is "local on the base scheme":

For example, from the internal perspective, the structure sheaf looks like an ordinary local ring (instead of a sheaf of rings with local stalks) and vector bundles look like ordinary free modules (instead of sheaves of modules satisfying a certain condition). The translation of internal statements and proofs is facilitated by an easy mechanical procedure.

The talk will give an introduction to this topic and show how the internal point of view can be exploited to give simpler definitions and more conceptual proofs of the basic notions and observations in algebraic geometry. We will also point out certain unexpected properties of the internal universe, for instance that the structure sheaf looks in fact even like a field.

(This is a report on my master's thesis [3].)

Outline

What is a topos?

Formal definition

A topos is a category which has finite limits, is cartesian closed and has a subobject classifier.

Motto

A topos is a category sufficiently rich to support an internal language.

Examples

- Set: category of sets
- Sh(X): category of set-valued sheaves on a space X

- While technically correct, the formal definition is actually misleading in a sense: A topos has lots of other vital structure, which is crucial for a rounded understanding, but is not listed in the definition (which is trimmed for minimality).

  A more comprehensive definition is: A *topos* is a locally cartesian closed, finitely complete and cocomplete Heyting category which is exact, extensive and has a subobject classifier.

- See [18] for a leisurely introduction to topos theory.

What is the internal language?

The internal language of a topos $\mathcal{E}$ allows to

1. construct objects and morphisms of the topos,
2. formulate statements about them and
3. prove such statements

in a naive element-based language:

external point of view	internal point of view
objects of $\mathcal{E}$	sets
morphisms of $\mathcal{E}$	maps of sets

Special case: The language of Set is the usual mathematical language.

- Actually, the objects of $\mathcal{E}$ feel more like *types* instead of *sets*: For instance, there is no global membership relation $\in$. Rather, for each object A of $\mathcal{E}$, there is a relation $\in_A : A \times \mathcal{P}(A) \to \Omega$, where $\mathcal{P}(A)$ is the power object of A and Ω is the object of truth values of $\mathcal{E}$ (can be understood as the power object of a terminal object).
- Compare with the embedding theorem for abelian categories: There, an explicit embedding into a category of modules is constructed. Here, we only change perspective and talk about the same objects and morphisms.
- There exists a weaker variant of the internal language which works in abelian categories. By using it, one can even pretend that the objects are abelian groups (instead of modules), and when constructing morphisms by appealing to the axiom of unique choice (which is a theorem), one doesn't even have to check linearity. The proof that this approach works uses only categorical logic (so ist mostly "just formal").
- For expositions of the internal language, see the books [15] (chapters D1 and D4) and [20] (chapter VI) or the lecture notes [27] (chapter 13).

The internal language of Sh(X)

Let X be a topological space. Then we can recursively define

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ.

$U \models f = g : \mathcal{F}$ $:\Longleftrightarrow$ $f|_U = g|_U \in \Gamma(U, \mathcal{F})$

$U \models \varphi \wedge \psi$ $:\Longleftrightarrow$ $U \models \varphi$ and $U \models \psi$

$U \models \varphi \vee \psi$ $:\Longleftrightarrow$ ~~$U \models \varphi$ or $U \models \psi$~~

there exists a covering $U = \bigcup_i U_i$ s. th. for all i:

$U_i \models \varphi$ or $U_i \models \psi$

$U \models \varphi \Rightarrow \psi$ $:\Longleftrightarrow$ for all open $V \subseteq U$: $V \models \varphi$ implies $V \models \psi$

$U \models \forall f : \mathcal{F}.\ \varphi(f)$ $:\Longleftrightarrow$ for all sections $f \in \Gamma(V, \mathcal{F})$, $V \subseteq U$: $V \models \varphi(f)$

$U \models \exists f : \mathcal{F}.\ \varphi(f)$ $:\Longleftrightarrow$ there exists a covering $U = \bigcup_i U_i$ s. th. for all i:

there exists $f_i \in \Gamma(U_i, \mathcal{F})$ s. th. $U_i \models \varphi(f_i)$

- The rules are called *Kripke-Joyal semantics* and can be formulated over any topos (not just sheaf topoi). They are not all arbitrary: Rather, they are very finely concerted to make the crucial properties about the internal language (see next slide) true.
- If $\mathcal{F}$ is an object of $\mathrm{Sh}(X)$, we write "$f : \mathcal{F}$" instead of "$f \in \mathcal{F}$" to remind us that $\mathcal{F}$ is not really (externally) a set consisting of elements, but that we only pretend this by using the internal language.
- There are two further rules concerning the constants $\top$ and $\bot$ (truth resp. falsehood):

$$\begin{aligned} U \models \top \quad &:\Longleftrightarrow \quad U = U \text{ (always fulfilled)} \\ U \models \bot \quad &:\Longleftrightarrow \quad U = \emptyset \end{aligned}$$

- Negation is defined as
$$\neg\varphi :\equiv (\varphi \Rightarrow \bot).$$
- The alternate definition "$U \models \varphi \vee \psi :\Leftrightarrow U \models \varphi$ or $U \models \psi$" would not be local (cf. next slide).

- Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then:

$$
\begin{aligned}
& X \models \ulcorner \alpha \text{ is injective} \urcorner \\
\iff & X \models \forall s, t : \mathcal{F}.\ \alpha(s) = \alpha(t) \Rightarrow s = t \\
\iff & \text{for all open } U \subseteq X, \text{ sections } s, t \in \Gamma(U, \mathcal{F}): \\
& \qquad U \models \alpha(s) = \alpha(t) \Rightarrow s = t \\
\iff & \text{for all open } U \subseteq X, \text{ sections } s, t \in \Gamma(U, \mathcal{F}): \\
& \qquad \text{for all open } V \subseteq U: \\
& \qquad\qquad \alpha_V(s|_V) = \alpha_V(t|_V) \text{ implies } s|_V = t|_V \\
\iff & \text{for all open } U \subseteq X, \text{ sections } s, t \in \Gamma(U, \mathcal{F}): \\
& \qquad \alpha_U(s|_U) = \alpha_U(t|_U) \text{ implies } s|_U = t|_U \\
\iff & \alpha \text{ is a monomorphism of sheaves}
\end{aligned}
$$

- The corner quotes "$\ulcorner \ldots \urcorner$" indicate that translation into formal language is left to the reader.

- Similarly, we have (exercise, use the rules!):

$$
\begin{aligned}
& X \models \ulcorner \alpha \text{ is surjective} \urcorner \\
\iff & X \models \forall s : \mathcal{G}.\ \exists t : \mathcal{F}.\ \alpha(t) = s \\
\iff & \alpha \text{ is an epimorphism of sheaves}
\end{aligned}
$$

- One can simplify the rules for often-occuring special cases:

$$U \models \forall s : \mathcal{F}.\ \forall t : \mathcal{G}.\ \varphi(s,t) \quad \Longleftrightarrow \quad \text{for all open } V \subseteq U, \text{ sections } s \in \Gamma(V,\mathcal{F}),\ t \in \Gamma(V,\mathcal{G}):\ V \models \varphi(s,t)$$

$$U \models \forall s : \mathcal{F}.\ \varphi(s) \Rightarrow \psi(s) \quad \Longleftrightarrow \quad \text{for all open } V \subseteq U, \text{ sections } s \in \Gamma(V,\mathcal{F}):\ V \models \varphi(s) \text{ implies } V \models \psi(s)$$

$$U \models \exists! s : \mathcal{F}.\ \varphi(s) \quad \Longleftrightarrow \quad \text{for all open } V \subseteq U, \text{ there is exactly one section } s \in \Gamma(V,\mathcal{F}) \text{ with: } V \models \varphi(s)$$

- One can extend the language to allow for *unbounded* quantification ($\forall A$ vs. $\forall a \in A$), see Shulman's stack semantics [25]. This is needed to formulate universal properties internal to $\mathrm{Sh}(X)$, for instance.
- One can further extend the language to be able to talk about locally internal categories over $\mathrm{Sh}(X)$ (in the sense of Penon [23], see for instance the appendix of [14]): Then one can do category theory internal to $\mathrm{Sh}(X)$ using the internal language, see [3].

  This specific approach is, as far as I am aware, original work. But of course, internal category theory has been done for a long time, see for instance the textbook [15]; also cf. [10].

The internal language of $\mathrm{Sh}(X)$

Crucial properties

- Locality: If $U = \bigcup_i U_i$, then $U \models \varphi$ iff $U_i \models \varphi$ for each i.
- Soundness: If $U \models \varphi$ and φ implies ψ constructively, then $U \models \psi$.

no $\varphi \vee \neg\varphi$, no $\neg\neg\varphi \Rightarrow \varphi$, no axiom of choice

Note: The internal logic of $\mathrm{Sh}(X)$ is classical (fulfils the law of excluded middle) iff X is discrete.

Examples

- $U \models f = 0$ iff $f|_U = 0 \in \Gamma(U, \mathcal{F})$.
- $U \models f = 0 \vee g = 0$ iff on a cover $U = \bigcup_i U_i$, $f|_{U_i} = 0$ or $g|_{U_i} = 0$.
- $U \models \neg\neg(f = 0)$ iff $f = 0$ holds on a dense open subset of U.

Why is constructive mathematics interesting?

- The internal logic of most topoi is constructive.
- From a constructive proof of a statement, it's always possible to mechanically extract an *algorithm* witnessing its truth. For example: A proof of the infinitude of primes gives rise to an algorithm which actually computes infinitely many primes (outputting one at a time, never stopping).
- By the celebrated *Curry–Howard correspondence*, constructive truth of a formula is equivalent to the existence of a program of a certain type associated to the formula.
- In constructive mathematics, one can experiment with (and draw useful conclusions also holding in a usual sense) *anti-classical dream axioms*, for instance the one of synthetic differential geometry:

  All functions $\mathbb{R} \to \mathbb{R}$ *are smooth.*

- Constructive accounts of classical theories are sometimes more elegant or point out some minor but interesting points which are not appreciated by a classical perspective.
- The philosophical question on the *meaning of truth* is easier to tackle in constructive mathematics.

Three rumours about constructive mathematics

1. There is a false rumour about constructive mathematics, namely that the term *contradiction* is generally forbidden. This is not the case, one has to distinguish between

   – a true proof by contradiction: "Assume φ were false. Then $\ldots$, contradiction. So φ is in fact true."

   which constructively is only a proof of the weaker statement $\neg\neg\varphi$, and

   – a proof of a negated formula: "Assume ψ were true. Then $\ldots$, contradiction. So $\neg\psi$ holds."

   which is a perfectly fine proof of $\neg\psi$ in constructive mathematics.

2. There is a similar rumour that constructive mathematicians *deny* the law of excluded middle. In fact, one can constructively prove that there is no counterexample to the law: For any formula φ, it holds that $\neg\neg(\varphi \vee \neg\varphi)$.

   In constructive mathematics, one merely doesn't *use* the law of excluded middle. (Only in concrete models, for example as provided by the internal universe of the sheaf topos on a non-discrete topological space, the law of excluded middle will actually be refutable.)

3. There is one last false rumour about constructive mathematics: Namely that most of mathematics breaks down in a constructive setting. This is only true if interpreted naively: Often, already very small changes to the definitions and statements (which are classically simply equivalent reformulations) suffice to make them constructively acceptable.

   In other cases, adding an additional hypotheses, which is classically always satisfied, is necessary (and interesting). Here is an example: In constructive mathematics, one can not show that any inhabited subset of the natural numbers possesses a minimal element. [One can also not show the negation – recall the previous false rumour.] But one can show (quite easily, by induction) that any inhabited and *detachable* subset of the natural numbers possesses a minimal element. A subset $U \subseteq \mathbb{N}$ is detachable iff for any number $n \in \mathbb{N}$, it holds that $n \in U$ or $n \notin U$.

   This has a computational interpretation: Given an arbitrary inhabited subset $U \subseteq \mathbb{N}$, one cannot algorithmically find its minimal element. But it *is* possible if one has an algorithmic *test of membership* for U.

See [4, 11, 28] for references about constructive mathematics. The blog [2] is also very informative.

The petit Zariski topos

Definition

The petit Zariski topos of a scheme X is the category $\mathrm{Sh}(X)$ of set-valued sheaves on X.

Basic look and feel

- Internally, the structure sheaf $\mathcal{O}_X$ looks like

  an ordinary ring.

- Internally, a sheaf of $\mathcal{O}_X$-modules looks like

  an ordinary module on that ring.

Sheaves of modules, internally

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

- $\mathcal{F}$ is locally of finite type iff internally, it is finitely generated:

$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \exists x_1, \ldots, x_n : \mathcal{F}.\ \forall x : \mathcal{F}.\ \exists a_1, \ldots, a_n : \mathcal{O}_X.\ x = \sum_i a_i x_i$$

- $\mathcal{F}$ is locally finitely free iff internally, it is a finitely free module:

$$\mathrm{Sh}(X) \models \bigvee_{n \geq 0} \ulcorner \mathcal{F} \cong \mathcal{O}_X^n \urcorner$$

- Similarly for finite presentation, coherence, flatness, …

Motto: We can understand notions of algebraic geometry as notions of linear algebra internal to $\mathrm{Sh}(X)$.

- $\mathcal{F}$ is *coherent* iff internally,

$$\begin{aligned}&\mathrm{Sh}(X) \models \ulcorner \mathcal{F} \text{ is finitely generated} \urcorner \wedge \\ &\quad \bigwedge_{n \geq 0} \forall f : \mathcal{O}_X^n \to \mathcal{F}.\ \ulcorner f \text{ is } \mathcal{O}_X\text{-linear} \urcorner \Rightarrow \ulcorner \ker f \text{ is finitely generated} \urcorner.\end{aligned}$$

- It's fruitful to build a *dictionary* between external and corresponding internal notions: After proving such an equivalence *once*, it can be used arbitrarily often (see next slide). In this way, much routine work can be avoided.

Proving in the internal universe

Well-known proposition

Let $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are locally of finite type, so is $\mathcal{F}$.

Proof

Follows at once from the following theorem of constructive linear algebra:

> *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of modules. If M' and M'' are finitely generated, so is M.*

Motto: We can understand basic statements of algebraic geometry as statements of linear algebra internal to $\mathrm{Sh}(X)$.

- Note that in the standard proof of the linear algebra fact, there is no actual *choice* of generators happening (which would require the axiom of choice, which is not available in the internal universe): One simply uses the elimination rule for the existential quantifier.
- Proving basic statements of algebraic geometry by interpreting linear algebra facts in the internal language of the petit Zariski topos is not only simpler than the standard approach, but also more *conceptual*.

The sheaf of rational functions

Classical definition

The sheaf K_X of rational functions on a scheme X is the sheafification of

$$U \subseteq X \longmapsto \Gamma(U, \mathcal{O}_X)[S(U)^{-1}],$$

where $S(U) = \{s \in \Gamma(U, \mathcal{O}_X) \mid s \in \mathcal{O}_{X,x} \text{ is regular for all } x \in U\}$.

Internal definition

K_X is the total quotient ring of $\mathcal{O}_X$.

Tensor products of sheaves of modules

Classical definition

The tensor product of $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$ is the sheafification of the presheaf

$$U \subseteq X \longmapsto \Gamma(U, \mathcal{F}) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{G}).$$

Internal definition

$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the usual tensor product of ordinary modules.

In a similar vein, exterior powers can be treated. This can be used to define (and work with) the Koszul complex in the internal universe.

The structure sheaf, internally

- Internally, $\mathcal{O}_X$ is a local ring:

$$\mathrm{Sh}(X) \models \forall x, y : \mathcal{O}_X.\ \ulcorner x + y \text{ inv.} \urcorner \Longrightarrow \ulcorner x \text{ inv. or } y \text{ inv.} \urcorner$$

- In fact, $\mathcal{O}_X$ is "almost" a field:

$$\mathrm{Sh}(X) \models \forall x : \mathcal{O}_X.\ \neg \ulcorner x \text{ inv.} \urcorner \Longrightarrow \ulcorner x \text{ is nilpotent} \urcorner$$

- The scheme X is reduced iff $\mathcal{O}_X$ is internally a reduced ring,

$$\mathrm{Sh}(X) \models \bigwedge_{n \geq 0} \forall x : \mathcal{O}_X.\ x^n = 0 \Longrightarrow x = 0,$$

iff $\mathcal{O}_X$ fulfils the field condition

$$\mathrm{Sh}(X) \models \forall x : \mathcal{O}_X.\ \neg \ulcorner x \text{ inv.} \urcorner \Longrightarrow x = 0.$$

- We write "$\mathrm{Sh}(X) \models \ldots$" instead of "$X \models \ldots$" to emphasize the particular topos we're using.
- As before, the corner quotes indicate that translation into formal language is left to the reader. For instance, we have:

$$\ulcorner x \text{ is invertible} \urcorner \quad :\equiv \quad \exists y : \mathcal{O}_X.\ xy = 1.$$

- The usual definition of a local ring (a nontrivial ring with exactly one maximal ideal) is equivalent to the more elementary condition given here – in classical logic. In constructive logic, the latter is much better behaved. (For instance, recall that one even needs Zorn's lemma to show the existence of a maximal ideal in a nontrivial ring.)
- In constructive mathematics, the notion of *field* bifurcates into several related but non-equivalent ones. A non-comprehensive list is the following:

$$\forall x.\, x = 0 \vee \ulcorner x \text{ inv.} \urcorner$$

$$\forall x.\, \neg \ulcorner x \text{ inv.} \urcorner \Rightarrow x = 0$$

$$\forall x.\, x \neq 0 \Rightarrow \ulcorner x \text{ inv.} \urcorner$$

Rank of a sheaf of modules

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module locally of finite type. There is a correspondence

upper semi-continuous functions on X	$\xleftrightarrow{\text{1-1 correspondence}}$	completed natural numbers
external		internal

under which the rank function of $\mathcal{F}$ maps to the minimal number of generators of $\mathcal{F}$.

Well-known proposition

Assume X to be reduced.
Then:

1. $\mathcal{F}$ is locally free iff its rank is locally constant.
2. $\mathcal{F}$ is locally free on a dense open subset.

Constructive linear algebra

Let M be a f. g. module over a field.
Then:

1. M is free iff the number above is an ordinary natural number.
2. M is always not not free.

- The proposition appears as “important hard exercise” (13.7.K) in Vakil’s excellent lecture notes [30].
- The proposition follows at once from the corresponding facts of linear algebra! The internal language machinery thus allows for a very simple and conceptual proof.

Proposition

If every inhabited subset of the natural numbers has a minimum, then the law of excluded middle holds. (So in constructive mathematics, one cannot prove the natural numbers to be complete in this sense.)

Proof

Let φ be an arbitrary formula. Define the subset

$$U := \{n \in \mathbb{N} \mid n = 1 \vee \varphi\} \subseteq \mathbb{N},$$

which surely is inhabited by $1 \in U$. So by assumption, there exists a number $z \in \mathbb{N}$ which is the minimum of U. We have

$$z = 0 \quad \vee \quad z > 0$$

(this is not obvious, but can be proven by induction).

If $z = 0$, we have $0 \in U$, so $0 = 1 \vee \varphi$, so φ holds.

If $z > 0$, then $\neg\varphi$ holds: Because if φ were true, zero would be an element of U, contradicting the minimality of z.

Proposition

The partially ordered set

$$\widehat{\mathbb{N}} := \{A \subseteq \mathbb{N} \mid A \text{ inhabited and upward closed}\}$$

is the least partially ordered set containing $\mathbb{N}$ and possessing minima of arbitrary inhabited subsets.

The embedding $\mathbb{N} \hookrightarrow \widehat{\mathbb{N}}$ is given by

$$n \in \mathbb{N} \longmapsto \uparrow(n) := \{m \in \mathbb{N} \mid m \geq n\}.$$

Proof

If $M \subseteq \widehat{\mathbb{N}}$ is an inhabited subset, its minimum is

$$\min M = \bigcup M \in \widehat{\mathbb{N}}.$$

The proof of the universal property is left to the reader.

External translation (see [22])

Let X be a topological space and consider the constant sheaf N with $\Gamma(U, N) = \{f : U \to \mathbb{N} \mid f \text{ continuous}\}$. Internally, the sheaf N plays the role of the ordinary natural numbers.

Then there is an 1-1 correspondence:

1. Let $A \hookrightarrow N$ be a subobject which is inhabited and upward closed from the internal point of view. Then

   $$x \longmapsto \inf\{n \in \mathbb{N} \mid n \in A_x\}$$

   is an upper semi-continous function on X.
2. Let $\alpha : X \to \mathbb{N}$ be a upper semi-continous function. Then

   $$U \subseteq X \longmapsto \{f : U \to \mathbb{N} \mid f \text{ continuous, } f \geq \alpha \text{ on } U\}$$

   is a subobject of N which internally is inhabited and upward closed.

Under the correspondence, locally constant functions map exactly to the [image of the] ordinary internal natural numbers [in the completed natural numbers].

- Here is an explicit example of a completed natural number which is not an ordinary natural number: Let $X = \operatorname{Spec} k[X]$ and $\mathcal{F} = k[X]/(X-a)^{\sim}$. The rank of $\mathcal{F}$ is 1 at a and zero otherwise. It corresponds to the internal completed natural number

$$\begin{aligned} z := {} & \min\{n \in \mathbb{N} \mid \ulcorner\mathcal{F} \text{ can be generated by } n \text{ elements}\urcorner\} = \\ & \min\{n \in \mathbb{N} \mid n \geq 1 \vee \ulcorner\text{the element } (X-a) \text{ of } \mathcal{O}_X \text{ is invertible}\urcorner\}. \end{aligned}$$

  We have the internal implications

$$\begin{aligned} \mathrm{Sh}(X) &\models \quad \ulcorner(X-a) \text{ inv.}\urcorner \Rightarrow z = 0 \\ \mathrm{Sh}(X) &\models \neg\ulcorner(X-a) \text{ inv.}\urcorner \Rightarrow z = 1, \end{aligned}$$

  but we do *not* have

$$\mathrm{Sh}(X) \models \ulcorner(X-a) \text{ inv.}\urcorner \ \vee\ \neg\ulcorner(X-a) \text{ inv.}\urcorner,$$

  which would imply

$$\mathrm{Sh}(X) \models z = 0 \vee z = 1,$$

  i. e. the false statement that $\mathcal{F}$ is locally free (of ranks 0 resp. 1).

- Here is a constructive proof of the first cited proposition in linear algebra: By assumption, the minimal number $n \in \mathbb{N}$ of generators for M exists. Let $x_1, \ldots, x_n$ be a generating family of minimal length n. We want to prove it to be linearly independent, so that it constitutes a basis.

  Let $\sum_i \lambda_i x_i = 0$. If any λ_i were invertible, the shortened family $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ would also generate M. By minimality of n, this is not possible. So each λ_i is not invertible. By the field assumption on A, it follows that each λ_i is zero.

- In constructive mathematics, one can not show that every finitely generated vector space over a field admits a finite basis. (Exercise: Proof this by showing that this would imply the law of excluded middle.) This is not because the space might strangely turn out to be infinite-dimensional, but merely because one may not be able to explicitly exhibit a finite basis.

- There is the so-called *negative translation*: To a zeroth approximation, this says that a formula φ is true classically iff $\neg\neg\varphi$ is true constructively. (Beware that this is a oversimplification! You can find the exact statement in [6].) By classical experience, we can therefore expect the statement *a finitely generated module over a field is not not free* to hold constructively, and a rigorous proof is easy as well.

Quasi-compactness

- Of course, quasi-compactness is not a local condition: A scheme can be covered by quasi-compact opens without itself being quasi-compact. Therefore there can *not* exist a formula φ such that

$$\mathrm{Sh}(X) \models \varphi \quad \Longleftrightarrow \quad X \text{ is quasi-compact},$$

  so quasi-compactness can not be characterized by the internal language.

- But quasi-compactness of X has consequences on the meta-properties of the internal language: The scheme X is quasi-compact iff for any directed set I and monotone family $(\varphi_i)_{i \in I}$ of formulas (i. e. $\mathrm{Sh}(X) \models \varphi_i \Rightarrow \varphi_j$ for $i \preceq j$), it holds that

$$\mathrm{Sh}(X) \models \bigvee_{i \in I} \varphi_i \quad \Longrightarrow \quad \text{there exists } i \in I \text{ such that } \mathrm{Sh}(X) \models \varphi_i.$$

- There are similar meta-characterizations of connectedness and locality (meaning that X possesses a focal point).

Well-known proposition

Let I be a directed set and $(\mathcal{F}_i)_{i \in I}$ be a directed family of $\mathcal{O}_X$-modules. If $\mathcal{F} := \operatorname{colim}_i \mathcal{F}_i$ is locally of finite type and X is quasi-compact, then there exists an index $i \in I$ such that the canonical map $\mathcal{F}_i \to \mathcal{F}$ is an epimorphism.

Proof

We have the following theorem of constructive linear algebra:

> *Let I be a directed set and $(M_i)_{i \in I}$ be a directed family of $\mathcal{O}_X$-modules. If $M := \operatorname{colim}_i M_i$ is finitely generated, then it holds that*
>
> $$\bigvee_{i \in I} \ulcorner \mathcal{F}_i \to \mathcal{F} \textit{ is surjective} \urcorner.$$

So the proposition follows from the characterization of quasi-compactness and the observation that the family of formulas

$$\ulcorner \mathcal{F}_i \to \mathcal{F} \text{ is surjective} \urcorner$$

is indeed monotone. (Internal directed colimits coincide with externally calculated ones.)

The gros Zariski topos

Definition

The gros Zariski topos $\mathrm{Zar}(S)$ of a scheme S is the category $\mathrm{Sh}(\mathrm{Sch}/S)$, i. e. it consists of certain functors $(\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow \mathrm{Set}$.

Basic look and feel

- For each S-scheme X, its functor of points $\underline{X}$ is an object of $\mathrm{Zar}(S)$. It feels like

  the set of points of X.

- Internally, $\underline{\mathbb{A}}^1_S$ (affine line), given by

$$\underline{\mathbb{A}}^1_S : X \longmapsto \Gamma(X, \mathcal{O}_X),$$

  looks like a field: $\mathrm{Zar}(S) \models \forall x : \underline{\mathbb{A}}^1_S.\ x \neq 0 \Longrightarrow \ulcorner x \text{ inv.} \urcorner$

- The overcategory Sch/S becomes a Grothendieck site by declaring families of jointly surjective open immersions to be covers. See for instance the excellent Stacks project [26] for details.
- Working in $\mathrm{Zar}(S)$ amounts to incorporating the philosophy of describing schemes by their functors of points into one's mathematical language.
- Explicitly, the functor $\underline{X}$ is given by $\underline{X}(T) = \mathrm{Hom}_S(T, X)$ for S-schemes T. Because the Zariski site is *subcanonical*, this functor is always a sheaf.
- The object $\underline{S}$ looks like an one-element set from the internal universe. This is to be expected.

- Hakim worked out a theory of schemes internal to topoi (but without using the internal language), see [13].
- The internal language of $\mathrm{Zar}(\operatorname{Spec} A)$ is related to Coquand's program about dynamical methods in algebra, see [7, 8, 9].
- The observation that $\underline{\mathbb{A}}^1_S$ is internally a field is due to Kock [16] (in the case $S = \operatorname{Spec} \mathbb{Z}$) and implies a curious meta-theorem:

  Because $\mathrm{Zar}(\operatorname{Spec} \mathbb{Z})$ is the *classifying topos* for the theory of local rings, any statement about local rings which is of a certain logical form holds for the *universal model* $\underline{\mathbb{A}}^1_{\operatorname{Spec} \mathbb{Z}}$ iff it holds for any local ring (in any universe, particularly Set).

  Therefore, in proving a statement of such a form about arbitrary local rings, one may assume that they even fulfil the field condition.

  There is a similar story for local A-algebras, see [1]. See [31] for a short exposition on the usefulness of classifying topoi and universal models.

- The affine line fulfils the axiom

$$\mathrm{Zar}(S) \models \ulcorner\text{every function } \underline{\mathbb{A}}^1_S \to \underline{\mathbb{A}}^1_S \text{ is a polynomial}\urcorner.$$

  Compare with the axiom of synthetic differential geometry:

$$\text{every function } \mathbb{R} \to \mathbb{R} \text{ is smooth.}$$

  See [17].

Group schemes

group scheme	internal definition	functor of points: $X \mapsto \ldots$
$\mathbb{G}_a$	$\underline{\mathbb{A}}^1_S$ (as additive group)	$\Gamma(X, \mathcal{O}_X)$
$\mathbb{G}_m$	$\{x : \underline{\mathbb{A}}^1_S \mid \ulcorner x \text{ inv.} \urcorner\}$	$\Gamma(X, \mathcal{O}_X)^\times$
μ_n	$\{x : \underline{\mathbb{A}}^1_S \mid x^n = 1\}$	$\{f \in \Gamma(X, \mathcal{O}_X) \mid f^n = 1\}$
GL_n	$\{M : {\underline{\mathbb{A}}^1_S}^{n \times n} \mid \ulcorner M \text{ inv.} \urcorner\}$	$\mathrm{GL}_n(\Gamma(X, \mathcal{O}_X))$

Motto: Internal to $\mathrm{Zar}(S)$, group schemes look like ordinary groups.

Open immersions

Definition

A formula φ is called

- decidable iff $\varphi \vee \neg\varphi$ holds.
- $\neg\neg$-stable iff $\neg\neg\varphi \Rightarrow \varphi$ holds.

We can similarly define a notion of open formulas in $\mathrm{Zar}(S)$.

Proposition

- For $f \in \Gamma(X, \mathcal{O}_X)$, the formula "$f(x) \neq 0$" is open.
- A morphism $p : Y \to X$ is an open immersion iff

$$\mathrm{Zar}(S) \models \ulcorner \underline{p} \text{ is injective and } ``x \in \mathrm{image}(\underline{p})" \text{ is open} \urcorner.$$

- A section $f \in \Gamma(X, \mathcal{O}_X)$ can internally be understood as an element of the set of all maps from $\underline{X}$ to $\underline{\mathbb{A}^1_S}$, because the global sections of the inner hom object $\mathcal{H}om(\underline{X}, \underline{\mathbb{A}^1_S})$ correspond to the global sections of $\mathcal{O}_X$.
- The morphism $\underline{p} : \underline{Y} \to \underline{X}$ is given by postcomposition with $p : Y \to X$.
- The notion of *open truth values* is treated in synthetic topology, see [12, 19].

Open problems

- Expand the dictionary between external and internal notions, in particular find internal characterizations for morphisms of schemes
  - being of finite type,
  - being separated,
  - being proper (related work: [21]).
- Find interesting and useful structure and properties of the internal universe for specific schemes, for instance projective n-space.
- Find general transfer principles for statements about A-modules M vs. $\mathcal{O}_{\operatorname{Spec} A}$-modules $M^\sim$.
- Study the internal universe of other gros topoi associated to a scheme (defined using more sophisticated Grothendieck sites). See [32].
- Contemplate about derived methods in the internal setting.
- Give an internal language of 2- or ∞-topoi, to be able to talk about derived schemes in an internal language (related work: [24], [29]).

Conclusion

Usual mathematics is…

formal manipulation of symbols	vs.	discovering exciting relationships

Mathematics internal to a topos is…

a technical device for simplifying routine proofs	vs.	exploring exciting alternative universes and their curious objects

…and can be fruitfully used in algebraic geometry

annotated version of these slides:
`http://xrl.us/gaeltopos`

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