On the tautological rings of $\mathcal{M}_{g,1}$ and its universal Jacobian

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We fix a base field of arbitrary characteristic. Denote by $\mathcal{M}_{g,1}$ the moduli space of smooth 1-pointed curves of genus $g \geq 2$, which is isomorphic to the universal curve over $\mathcal{M}_g$. We shall work in the following setting.

\[
\begin{array}{c}
\mathcal{C} \\ p \\ \downarrow \\
\mathcal{M}_{g,1} \\
\end{array}
\quad \begin{array}{c}
\quad \leftarrow \\
\quad \downarrow \\
\quad \mathcal{J} \\
\quad \leftarrow \\
\quad \sigma_0
\end{array}
\]

Here $\mathcal{C}$ (resp. $\mathcal{J}$) is the universal curve (resp. universal Jacobian) over $\mathcal{M}_{g,1}$. The map $\pi: \mathcal{M}_{g,1} \to \mathcal{J}$ is the embedding induced by $x_0$.

### Moduli side

Denote by $K$ the first Chern class of the relative dualizing sheaf of $p$. We define classes

\[\kappa_i := p_i(K^{i+1}) \in CH_0^g(\mathcal{M}_{g,1}), \text{ for } i \geq 0,\]

\[\psi := x_0^*(K) \in CH_0^1(\mathcal{M}_{g,1}),\]

and we define the tautological ring $\mathcal{R}(\mathcal{M}_{g,1})$ to be the $Q$-subalgebra of $CH(\mathcal{M}_{g,1})$ generated by the classes above. Faber made the following conjectures.

(i) The ring $\mathcal{R}(\mathcal{M}_{g,1})$ is Gorenstein with total dimension $g - 3$. It means that $\mathcal{R}^i(\mathcal{M}_{g,1}) = 0$ for $i > g - 1$, that $\mathcal{R}^{g-1}(\mathcal{M}_{g,1}) \simeq Q$, and that the natural paring between $\mathcal{R}^i(\mathcal{M}_{g,1})$ and $\mathcal{R}^{g-1-i}(\mathcal{M}_{g,1})$ is perfect for all $0 \leq i \leq g - 1$.

(ii) The ring $\mathcal{R}(\mathcal{M}_{g,1})$ is generated by $\kappa_1, \ldots, \kappa_{[g/3]}$ and $\psi$. There are no relations between these classes in codimension $\leq [g/3]$.

The difficulty of proving these conjectures is to find sufficiently many relations between tautological classes.

### Jacobian side

We define the tautological ring $\mathcal{T}(\mathcal{J})$ to be the smallest $Q$-subalgebra of $CH(\mathcal{J})$ that contains $[C] := [i(C)] \in CH^{-1}_0(\mathcal{J})$, and that is stable under the Fourier transform $\mathcal{F}$ and the Beauville decomposition. Consider the Beauville decomposition of $[C]$

\[[C] = \sum_{j=0}^{2g-2} [C]_{(i,j)} \text{ with } [C]_{(i,j)} \in CH_{i,j}^{-1}(\mathcal{J}).\]

Define $\theta := -\mathcal{F}([C]_{(0)}) \in CH_{(0)}^{-1}(\mathcal{J})$. Polishchuk [Pol07] proves that the operators

\[e(\alpha) := -\theta \cdot \alpha,\]

\[f(\alpha) := -\mathcal{F}([C]_{(0)} \ast \alpha),\]

\[h(\alpha) := (2i - j - g) \cdot \alpha, \text{ for } \alpha \in CH_{i,j}(\mathcal{J})\]

generate an $sl_2$-action on $CH_0(\mathcal{J})$ (resp. $\mathcal{T}(\mathcal{J})$).

For $0 \leq j \leq 2g - 2$ and $j/2 \leq i \leq j + 1$, we define the classes

\[p_i^j := F(\theta^{i-j} \cdot [C]_{(i)}) \in \mathcal{T}_{ij}(\mathcal{J}).\]

Using Polishchuk’s results, we prove that $\mathcal{T}(\mathcal{J})$ is generated by $p_i^j$ and $\psi := \pi^*(\psi)$. We also show that $f \in sl_2$ acts on $\mathcal{T}(\mathcal{J})$ via an explicit degree 2 differential operator $\mathcal{D}$. Further, the ring $\mathcal{R}(\mathcal{M}_{g,1})$ can be identified as a $Q$-subalgebra of $\mathcal{T}(\mathcal{J})$.

More importantly, we obtain a powerful method to produce relations in $\mathcal{T}(\mathcal{J})$ (resp. $\mathcal{R}(\mathcal{M}_{g,1})$): take any polynomial in $p_i^j$ and $\psi$ that vanishes for trivial (motivic) reasons, then apply the operator $\mathcal{D}$ one or several times. The resulting polynomial should vanish as well. In this way we get a huge space of ‘obvious’ relations that are simply dictated by the $sl_2$-action.

### Main results

Using our relations, we prove the following main results.

(i) The ring $\mathcal{R}(\mathcal{M}_{g,1})$ is generated by $\kappa_1, \ldots, \kappa_{[g/3]}$ and $\psi$. By pushing forward to $\mathcal{M}_g$, we also obtain that $\mathcal{R}(\mathcal{M}_g)$ is generated by $\kappa_1, \ldots, \kappa_{[g/3]}$. This gives a new proof of part of Faber’s conjectures, which was first obtained by Kollár [Kol05].

(ii) Computation confirms that Faber’s conjectures for $\mathcal{M}_{g,1}$ are true for $g \leq 19$. From $g = 20$ on, our relations do not produce Gorenstein rings.

(iii) By pushing forward to $\mathcal{M}_g$, we obtain a new proof of Faber’s conjectures (for $\mathcal{M}_g$) for $g \leq 23$. For $g \geq 24$, computation gives the same relations as the Faber-Zagier computations.

(iv) We also give an algebraic proof of an identity obtained by Morita (cf. [HR01]):

\[\pi_*([C]_{(i)} \ast F([C]_{(i)})) = \kappa_1/6 + g \psi \in \mathcal{R}^1(\mathcal{M}_{g,1}).\]

Beyond all these results, our approach has many advantages compared to previous ones. From a theoretical perspective, it gives an extremely clean and uniform treatment of Faber’s conjectures, which converts a geometric problem into a combinatorial problem. In this way, many complicated facts become obvious. On the practical side, our method produces huge quantities of relations, and is also very computer-friendly.

Finally, the nature of our approach (using the $sl_2$-action as source of relations) also suggests that these might be the only relations we can ever find. There has been some work in this direction as well.
References


[Yin12] Q. Yin, On the tautological rings of \( \mathcal{M}_{g,1} \) and its universal Jacobian. Preprint 2012.