On the tautological rings of $\mathcal{M}_{g,1}$ and its universal Jacobian

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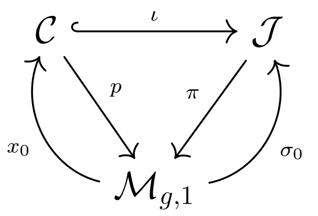
WE fix a base field of arbitrary characteristic. Denote by $\mathcal{M}_{g,1}$ the moduli space of smooth 1-pointed curves of genus $g \geq 2$, which is isomorphic to the universal curve over $\mathcal{M}_g$. We shall work in the following setting.



Here $\mathcal{C}$ (resp. $\mathcal{J}$) is the universal curve (resp. universal Jacobian) over $\mathcal{M}_{g,1}$. The map x_0 (resp. σ_0) is the section given by the marked point (resp. zero section), and ι is the embedding induced by x_0.

Moduli side

DENOTE by K the first Chern class of the relative dualizing sheaf of p. We define classes

$$\kappa_i := p_*(K^{i+1}) \in \mathrm{CH}^i_{\mathbb{Q}}(\mathcal{M}_{g,1}), \text{ for } i \geq 0,$$
$$\psi := x_0^*(K) \in \mathrm{CH}^1_{\mathbb{Q}}(\mathcal{M}_{g,1}),$$

and we define the *tautological ring* $\mathcal{R}(\mathcal{M}_{g,1})$ to be the $\mathbb{Q}$-subalgebra of $\mathrm{CH}_{\mathbb{Q}}(\mathcal{M}_{g,1})$ generated by the classes above. Faber made the following conjectures.

(i) The ring $\mathcal{R}(\mathcal{M}_{g,1})$ is Gorenstein with socle in codimension $g-1$. It means that $\mathcal{R}^i(\mathcal{M}_{g,1}) = 0$ for $i > g-1$, that $\mathcal{R}^{g-1}(\mathcal{M}_{g,1}) \simeq \mathbb{Q}$, and that the natural paring between $\mathcal{R}^i(\mathcal{M}_{g,1})$ and $\mathcal{R}^{g-1-i}(\mathcal{M}_{g,1})$ is perfect for all $0 \leq i \leq g-1$.

(ii) The ring $\mathcal{R}(\mathcal{M}_{g,1})$ is generated by $\kappa_1, \ldots, \kappa_{\lfloor g/3 \rfloor}$ and ψ. There are no relations between these classes in codimension $\leq \lfloor g/3 \rfloor$.

The difficulty of proving these conjectures is to find sufficiently many relations between tautological classes.

Jacobian side

WE define the *tautological ring* $\mathcal{T}(\mathcal{J})$ to be the smallest $\mathbb{Q}$-subalgebra of $\mathrm{CH}_{\mathbb{Q}}(\mathcal{J})$ that contains $[\mathcal{C}] := \big[\iota(\mathcal{C})\big] \in \mathrm{CH}^{g-1}_{\mathbb{Q}}(\mathcal{J})$, and that is stable under the Fourier transform $\mathcal{F}$ and the Beauville decomposition.

Consider the Beauville decomposition of $[\mathcal{C}]$

$$[\mathcal{C}] = \sum_{j=0}^{2g-2} [\mathcal{C}]_{(j)} \text{ with } [\mathcal{C}]_{(j)} \in \mathrm{CH}^{g-1}_{(j)}(\mathcal{J}).$$

Define $\theta := -\mathcal{F}\big([\mathcal{C}]_{(0)}\big) \in \mathrm{CH}^1_{(0)}(\mathcal{J})$. Polishchuk [Pol07] proves that the operators

$$e(\alpha) := -\theta \cdot \alpha,$$
$$f(\alpha) := -[\mathcal{C}]_{(0)} * \alpha,$$
$$h(\alpha) := (2i - j - g) \cdot \alpha, \text{ for } \alpha \in \mathrm{CH}^i_{(j)}(\mathcal{J})$$

generate an $\mathfrak{sl}_2$-action on $\mathrm{CH}_{\mathbb{Q}}(\mathcal{J})$ (resp. $\mathcal{T}(\mathcal{J})$).

For $0 \leq j \leq 2g-2$ and $j/2 \leq i \leq j+1$, we define the classes

$$p^i_j := \mathcal{F}\big(\theta^{j-i+1} \cdot [\mathcal{C}]_{(j)}\big) \in \mathcal{T}^i_{(j)}(\mathcal{J}).$$

Using Polishchuk's results, we prove that $\mathcal{T}(\mathcal{J})$ is generated by p^i_j and $\psi := \pi^*(\psi)$. We also show that $f \in \mathfrak{sl}_2$ acts on $\mathcal{T}(\mathcal{J})$ via an explicit degree 2 differential operator $\mathcal{D}$. Further, the ring $\mathcal{R}(\mathcal{M}_{g,1})$ can be identified as a $\mathbb{Q}$-subalgebra of $\mathcal{T}(\mathcal{J})$.

More importantly, we obtain a powerful method to produce relations in $\mathcal{T}(\mathcal{J})$ (resp. $\mathcal{R}(\mathcal{M}_{g,1})$): take any polynomial in p^i_j and ψ that vanishes for trivial (motivic) reasons, then apply the operator $\mathcal{D}$ one or several times. The resulting polynomial should vanish as well. In this way we get a huge space of 'obvious' relations that are simply dictated by the $\mathfrak{sl}_2$-action.

Main results

USING our relations, we prove the following main results.

(i) The ring $\mathcal{R}(\mathcal{M}_{g,1})$ is generated by $\kappa_1, \ldots, \kappa_{\lfloor g/3 \rfloor}$ and ψ. By pushing forward to $\mathcal{M}_g$, we also obtain that $\mathcal{R}(\mathcal{M}_g)$ is generated by $\kappa_1, \ldots, \kappa_{\lfloor g/3 \rfloor}$. This gives a new proof of part of Faber's conjectures, which was first obtained by Ionel [Ion05].

(ii) Computation confirms that Faber's conjectures for $\mathcal{M}_{g,1}$ are true for $g \leq 19$. From $g = 20$ on, our relations do not produce Gorenstein rings.

(iii) By pushing forward to $\mathcal{M}_g$, we obtain a new proof of Faber's conjectures (for $\mathcal{M}_g$) for $g \leq 23$. For $g \geq 24$, computation gives the same relations as the Faber-Zagier relations.

(iv) We also give an algebraic proof of an identity obtained by Morita (*cf.* [HR01]):

$$\pi_*\Big([\mathcal{C}]_{(1)} \cdot \mathcal{F}\big([\mathcal{C}]_{(1)}\big)\Big) = \kappa_1/6 + g\psi \in \mathcal{R}^1(\mathcal{M}_{g,1}).$$

Beyond all these results, our approach has many advantages compared to previous ones. From a theoretical perspective, it gives an extremely clean and uniform treatment of Faber's conjectures, which converts a geometric problem into a combinatorial problem. In this way, many complicated facts become obvious. On the practical side, our method produces huge quantities of relations, and is also very computer-friendly.

Finally, the nature of our approach (using the $\mathfrak{sl}_2$-action as source of relations) also suggests that these might be the only relations we can ever find. There has been some work in this direction as well.

References

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