Stacks of ramified Galois covers

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Galois covers

LET G be a flat and finite group scheme. A Galois cover of group G (or simply G-cover) of Y is a cover $X \xrightarrow{\rho} Y$ together with an action of G on X such that the G-representations $\rho_* \mathcal{O}_X$ and $\mathcal{O}_Y[G]$ are (fppf) locally isomorphic as representations. For istance any G-torsor is a G-cover, as any cover $X \xrightarrow{\rho} Y$ between integral varieties with an action of G on X over Y such that $X/G \simeq Y$, provided some assumptions on the characteristic. The problem I have looked at is to understand better these covers and the geometry of the moduli stack G-Cov of G-covers. When $G = \mu_2$ a G-cover of Y is simply given by an invertible sheaf $\mathcal{L}$ and a section of $\mathcal{L}^2$, while when $G = \mu_3$ is given by invertible sheaves $\mathcal{L}_1, \mathcal{L}_2$ and maps $\mathcal{L}_1^2 \longrightarrow \mathcal{L}_2$, $\mathcal{L}_2^2 \longrightarrow \mathcal{L}_1$. A similar description also exists for $G = \mu_2^2$. Beyond those cases very little is known, except for a description, given in [Par91], of abelian covers of smooth varieties whose total space is normal in terms of invertible sheaves.

Abelian case

The first case I have considered is the abelian one, i.e. when G is a diagonalizable group scheme over $\mathbb{Z}$ with dual $M = \text{Hom}(G, \mathbb{G}_m)$. The geometry of G-Cov, as the description of G-covers, grows more intricate as $|M|$ increases. Indeed G-Cov is smooth (and irreducible) if and only if $G = \mu_2, \mu_3, (\mu_2)^2$, and it is reducible if $|M| \geq 9$. At least it is always connected over any field. Its geometric nature is deeply connected to the one of the equivariant Hilbert schemes Hilb M, that parametrize particular M-equivariant closed subschemes of some $\mathbb{A}^r$. For istance G-Cov has a principal irreducible component $\mathcal{Z}_G$, which is the closure of G-torsors. Generalizing some ideas of [Par91] it is possible to describe certain families of $\mathcal{Z}_G$ in terms of invertible sheaves and other combinatorial data, obtaining in particular a description of its smooth locus $\mathcal{Z}_G^{sm}$, which turns out to be a toric stack, and a description of G-covers of smooth varieties whose total space is normal crossing in codimension 1.

Non-Abelian case

The property that makes the abelian case easier to work out with respect to the general one is that the representation theory is very simple, since all the irreducible representations are one dimensional and, consequently, we only have to deal with invertible sheaves. This is clearly no longer true in general. Say G is a usual finite group and assume to work over an algebraically closed field of characteristic prime to $|G|$. In terms of sheaves, a G-cover of Y can be described as an additive functor $\mathcal{E}$ from the category of representations of G to the category of locally free sheaves of Y such that $\text{rk}\, \mathcal{E}_V = \dim V$ for any (irreducible) representation of G, $\mathcal{E}_B = \mathcal{O}_Y$, plus a 'monoidal' structure, i.e. maps $\mathcal{E}_V \otimes \mathcal{E}_W \longrightarrow \mathcal{E}_{V \otimes W}$ for any pair of representations V, W with obvious compatibility conditions. G-torsors correspond to the case when all those maps are isomorphisms.

- $\mathbf{G = S_3}$. Its irreducible representations are B, A, V, where B is the trivial one, A is the alternating one, while V is the 2 dimensional one. So we will need an invertible sheaf $\mathcal{E}_A = \mathcal{L}$, a rank 2 locally free sheaf $\mathcal{E}_V = \mathcal{F}$ and maps $\mathcal{L}^2 \longrightarrow \mathcal{O}_Y$, $\mathcal{L} \otimes \mathcal{F} \longrightarrow \mathcal{F}$ and $\mathcal{F}^2 \longrightarrow \mathcal{O}_Y \oplus \mathcal{L} \oplus \mathcal{F}$. Compatibility conditions yield some independent commutative diagrams between those maps that cannot be simplified and it turns out that S_3-Cov is not reduced and has 2 irreducible components. On the other hand, if Y is a smooth variety, it is possible to describe S_3-covers $X \longrightarrow Y$ such that X is smooth. They all come from particular degree 3 covers $X' \longrightarrow Y$, with X' smooth, that satisfy certain conditions in codimension 2. By [Mir85], a degree 3 cover of Y is given by a rank 2 sheaf $\mathcal{F}$ and a section of $V = H^0((\text{Sym}^3 \mathcal{F})^\vee \otimes \det \mathcal{F})$. If Y is a surface and $\mathcal{F}$ is 'ample' enough, the (open) locus of V of S_3-covers whose total space is smooth is not empty and we can compute the standard invariants of the surfaces obtained in this way.
- $\mathbf{G = D_n}$. Although its representation theory is still very simple, the compatibility conditions become very difficult to understand. On the other hand it is still possible to describe D_n-covers of smooth varieties whose total space is regular in codimension 1, but the data needed to build them can not be obtained easily, even if we are working on simple varieties as $\mathbb{P}^2$.
- **G =??.** It seems very hard to obtain a unifying and explicit description of G-covers. More reasonably one can try to build families of G-covers, i.e. describe particular loci of G-Cov, or focus on covers whose total space is regular.

References

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[Par91] Rita Pardini. Abelian covers of algebraic varieties. *J. reine angew. Math.*, 417:191–213, 1991.

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