

Moduli of parabolic sheaves

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The aim of my current research is to generalize some known results about moduli of parabolic sheaves on a projective variety with an effective divisor ([3]) to the case of a more general logarithmic scheme.

Parabolic sheaves

Parabolic sheaves were introduced to generalize a correspondence between representations of the fundamental group of a projective curve and a certain type of vector bundles to the non-proper case, and the notion was later adapted to more general settings. On a projective variety X with an effective (smooth) cartier divisor D (for example a curve with a marked point), the definition is as follows:

Definition 1. ([3]) *A parabolic sheaf on X with respect to D is a (torsion-free) quasi-coherent sheaf E on X , together with a filtration*

$$E(-D) = E_{d+1} \subset E_d \subset \cdots \subset E_2 \subset E_1 = E$$

and real numbers (called weights) $0 \leq \alpha_1 < \cdots < \alpha_d < 1$.

One can define a “parabolic Hilbert polynomial” and a notion of (semi-)stability for parabolic sheaves, and carry on the construction of a moduli space of (semi-)stable parabolic sheaves with fixed rational weights and Hilbert polynomial, in the spirit of the classical theory of moduli of coherent sheaves ([2]). The result is:

Theorem 2. ([3]) *There is a projective variety parametrizing (S -equivalence classes of) parabolic semi-stable sheaves on X with respect to D , with fixed rational weights and parabolic Hilbert polynomial. Moreover, there is an open subscheme that parametrizes parabolic stable sheaves.*

The divisor D above defines what is called a “logarithmic structure” on X , and the notion of parabolic sheaf generalizes to the setting of logarithmic schemes ([1]).

Stacks of roots

Parabolic sheaves with rational weights can be described as quasi-coherent sheaves on an algebraic stack, called a root stack.

Given a projective variety X with a smooth cartier divisor D and an integer n , there is an algebraic stack $X_n \rightarrow X$ over X that “classifies” n -th roots of the pair $(\mathcal{O}(D), s)$, where s is the canonical section of the line bundle $\mathcal{O}(D)$, i.e. pairs (L, t) with L an invertible sheaf and t a global section, such that $(L, t)^{\otimes n} \simeq (\mathcal{O}(D), s)$.

This root construction can be applied to more general logarithmic schemes, and one has:

Theorem 3. ([1]) *The category of parabolic sheaves on a logarithmic scheme X with weights with denominator dividing n is equivalent to the category of quasi-coherent sheaves on the root stack X_n .*

My research

This last result allows to apply a theory of moduli of coherent sheaves on an algebraic stack, which was developed by Nironi in [4], to get a moduli space for parabolic sheaves with fixed denominators.

Theorem 4. *Given a projective, fine and saturated logarithmic scheme X with a system of denominators $A \subseteq B$ that admits a global chart, there exists a projective variety parametrizing (S -equivalence classes of) parabolic coherent sheaves on X . The same is true if instead of a chart we have a “locally constant sheaf of charts”.*

The point is that these data allow to build a “generating sheaf” (needed for Nironi’s machinery to work), that depends on the chart or the sheaf of charts. The resulting stability and moduli spaces seem to depend on the choice of this piece of data too, and one could expect to get interesting maps between the moduli spaces by varying it, as happens by varying polarizations in the classical theory, for example.

The next step is to allow the weights to vary, and to try to obtain a moduli space of parabolic (semi-)stable sheaves with arbitrary denominators. From this side of the problem, it is natural to consider the inverse limit $X_\infty = \varprojlim_n X_n$: morally, parabolic sheaves with arbitrary weights should correspond to quasi-coherent sheaves on X_∞ , and moreover this “pro-algebraic” stack seems to control much of the geometry of the logarithmic scheme X .

References

- [1] N. Borne and A. Vistoli, *Parabolic sheaves on logarithmic schemes*, <http://arxiv.org/abs/1001.0466>.
- [2] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Cambridge University Press.
- [3] M. Maruyama and K. Yokogawa, *Moduli of parabolic stable sheaves*, *Mathematische Annalen* (1991).
- [4] F. Nironi, *Moduli spaces of semistable sheaves on projective deligne-mumford stacks*, <http://arxiv.org/abs/0811.1949>.