# Noncommutative Motives

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## Introduction

NON-COMMUTATIVE GEOMETRY is more of a principle than a subject: if $X$ is an object of some sort of geometrical nature, our tendency is to encode the geometrical information of $X$ into some simpler mathematical object $T_X$ which, of course, depends on what kind of information we want to capture. Then, by isolating the formal properties of these objects, we think of them as *non-commutative spaces/schemes*.
In algebraic geometry, the natural geometric objects are schemes and the traditional choice for $T_X$ is the category $Qcoh(X)$ of quasi-coherent sheaves on $X$. This is mostly because all the typical information we want to extract ($K$-theory, cohomology groups, etc) can actually be read from this structure. It was Grothendieck who soon foresaw that we should be able to extract all this information from the more flexible homotopy theory of complexes of quasi-coherent sheaves. Unfortunately, in the old days people were only working with a small part of this theory - the one encoded in the derived category. Technical aspects blocked the way until, in the 90's, dg-categories appeared in the game providing a technological enhancement to the first notion. Finally the initial vision regain a new hope [1]. In his works [1], Kontsevich initiated a research program promoting dg-categories (or their homotopy equivalent brothers - the $A_\infty$-categories) as an ultimate notion of non-commutative space. Not only this, he also understood that, similarly to schemes, the non-commutative side should also admit a motivic theory [5].
Every scheme $X$ gives rise to a $k$-dg-category $L_{qcoh}(X)$ - *the dg-derived category of* $X$. Moreover, this assignment can be properly understood as a map

$$\text{Classical Schemes} \rightarrow \text{NC-Schemes}$$

Not only schemes give rise to nc-schemes. Many other different types of mathematical objects can be used as an input: algebras ($T_X$ = the dg-derived category of complexes of $X$-modules), differential graded algebras, symplectic manifolds ($T_X$= Fukaya dg-category of $X$), complex varieties, problems in deformation quantization, etc. It follows that one of the main philosophical advantages of the theory is that we can bring different kinds of geometrical objects together in the same world and treat them as equal.

## My Research

IN the late 90's, V.Voevodsky and F. Morel developed an homotopy theory for schemes, together with a stabilized motivic version [6]. The idea was to mimic the classical stable homotopy theory of spaces and therefore provide a setting where both schemes and their cohomology theories can be treated in equal terms. Their construction was performed using the techniques of model category theory. Nowadays we know that a model category is a mere strict presentation of a more fundamental object - an $\infty$-category. Every model category has an underlying $\infty$-category and the later is what really matters. It is important to say that the need for this passage overcomes the philosophical reasons. Thanks to the techniques of [2, 3] we have the ways to do and prove things which would forever be out of range only with the highly restrictive techniques of model categories.

The first part of my research project concerned the universal characterization of the $\infty$-category underlying the original construction of Voevodsky-Morel, with its symmetric monoidal structure.

**Theorem**([4]): The canonical map from the category of schemes to the $(\infty,1)$-category underlying the stable motivic homotopy theory of schemes is the universal functor satisfying the following properties: $(i)$ has values in a stable presentable symmetric monoidal $(\infty,1)$-category; $(ii)$ is monoidal with respect to the cartesian product of schemes;$(iii)$ satisfies Nisnevich descent, $\mathbb{A}^1$-invariance and the image of $\mathbb{P}^1$ mod out by the image of the point at $\infty$ becomes invertible.

This characterization becomes meaningful if we want to compare the motives of schemes with the motives of something else. Of course, the something else we have in mind are the non-commutative schemes. We can give a sense to the constructions of Voevodsky-Morel in the non-commutative world and for free, we have the following result:

**Corollary:** There is a unique monoidal dotted arrow at the motivic level, completing the diagram in a commutative way

$$\begin{tikzcd}
\text{Classical Schemes} \arrow[r] \arrow[d] & \text{NC-Schemes} \arrow[d] \\
\text{Stable Motivic Homotopy} \arrow[r, dashed] & \text{NC-Stable Motivic Homotopy}
\end{tikzcd}$$

We emphasize the "free" aspect. In general, this kind of comparison map is extremely hard to obtain only with constructive methods and the techniques of model category theory.

Another important future application concerns the link between the classical and the noncommutative Hodge Structures (see Anthony Blanc's poster).

This work has been carried out during this year and a preprint is now available in the arxiv. The next step (the important one) is the study of the comparison map itself. It is known that the original map from schemes to nc-schemes is far from being injective or surjective. But what happens at the motivic level?

## References

[1] M. Kontsevich. *Mixed noncommutative motives - Talk at the Workshop on Homological Mirror Symmetry. University of Miami. 2010. Notes available at www-math.mit.edu/auroux/frg/miami10-notes*.

[2] J. Lurie. *Higher Topos Theory http://www.math.harvard.edu/ lurie/papers/highertopoi.pdf*. Annals of Mathematics Studies, 2009.

[3] J. Lurie. *Higher Algebra http://www.math.harvard.edu/ lurie/papers/higheralgebra.pdf*. February 2012.

[4] M. Robalo. *Noncommutative Motives I: A Universal Characterization of the Motivic Stable Homotopy Theory of Schemes*. arxiv.

[5] G. Tabuada. *A Guided Tour Through the Garden of Noncommutative Motives*. Clay Mathematics Proceedings Volume 17, 2012.

[6] V. Voevodsky and F. Morel. $\mathbb{A}^1$*-Homotopy Theory of Schemes*. Publications Mathématiques de l'IHÉS 90, 1999.

$*$ Supported by the Portuguese Foundation for Science and Technology - FCT Project SFRH / BD / 68868 / 2010
[1] Obi-Wan had nothing to do with this!