1-Motives and Independence of ℓ

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Introduction

We start with the following warm-up question:

Question. Let X be a smooth, possibly non-proper variety over an algebraically closed field k. What geometric structure does the Picard group $\mathrm{Pic}(X)$ possess?

When X is proper, then $\mathrm{Pic}(X)$ contains a subgroup $\mathrm{Pic}^0(X)$ which is the k-points of an abelian variety. When X is not proper we do not have this. But suppose we can compactify X to a smooth proper variety $\overline{X}$ with complement $D := \overline{X} - X$ a divisor with strict normal crossings (e.g., if $\mathrm{char}\, k = 0$). Then we can define a subgroup $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ by restricting the Weil divisor exact sequence to $\mathrm{Pic}^0(\overline{X})$, yielding

$$\mathrm{Div}^0_D(\overline{X}) \to \mathrm{Pic}^0(\overline{X}) \to \mathrm{Pic}^0(X) \to 0,$$

where $\mathrm{Div}^0_D(\overline{X})$ is the divisors on $\overline{X}$ supported on D which land in $\mathrm{Pic}^0(\overline{X})$. The subgroup $\mathrm{Pic}^0(X)$ can be shown to be independent of the choice of compactification. This suggests that the correct geometric structure to put on $\mathrm{Pic}^0(X)$ is that of a 1-motive:

Definition. A 1-motive over an algebraically closed field k consists of the following data:

1. a semiabelian variety G (i.e., an extension of an abelian variety by a torus),
2. a lattice L (free, finitely generated abelian group), and
3. a map $L \to G$.

If k is not algebraically closed we require all of these data to be defined over k in a suitable sense.

The construction above produces a 1-motive $M^1(X) := [\mathrm{Div}^0_D(\overline{X}) \to \mathrm{Pic}^0(\overline{X})]$ associated to a smooth, non-proper variety X. One can define a notion of isomorphism of 1-motives, and then the motive $M^1(X)$ is independent of the choice of $\overline{X}$ [BVS].

1-Motives as universal cohomology

Recall that an algebraic variety X over a field k has several associated cohomology groups which are 'topological' in nature. Foremost among these are singular cohomology $H^*_{\mathrm{sing}}(X, \mathbb{Q})$ if $k \subseteq \mathbb{C}$ and étale cohomology $H^*_{\mathrm{et}}(X_{\overline{k}}, \mathbb{Q}_\ell)$ for any field k and any $\ell \neq \mathrm{char}\, k$. One also has variants $H^*_{c,\mathrm{sing}}(X, \mathbb{Q})$ and $H^*_{c,\mathrm{et}}(X, \mathbb{Q}_\ell)$ for compactly supported cohomology. A primary purpose of motives is to be 'universal' for these various cohomology theories. For 1-motives this takes the form of realization functors

$$T_{Hodge} : 1 - Mot(k) \to MHS(k),$$
$$T_\ell : 1 - Mot(k) \to \mathrm{Rep}_{\mathrm{Gal}(\overline{k}/k)}(\mathbb{Q}_\ell)$$

to the category of mixed Hodge structures and ℓ-adic Galois representations, respectively [BVS]. In our example above, the 1-motive $M^1(X)$ in fact has the property that $T_{Hodge}(M^1(X)) \cong H^1(X, \mathbb{Q})$ with its mixed Hodge structure, and $T_\ell(M^1(X)) \cong H^1(X, \mathbb{Q}_\ell)$ for prime $\ell \neq \mathrm{char\, k}$, and this determines $M^1(X)$ up to isomorphism. In general one can pose the following problem:

Problem. For a d-dimensional variety X over a perfect field k (possibly singular and non-proper), define 1-motives $M^1(X)$, $M^1_c(X)$, $M^{2d-1}(X)$, $M^{2d-1}_c(X)$ realizing the corresponding cohomology and compactly supported cohomology groups of a variety over X. These 1-motives should be unique up to isomorphism and contravariantly functorial for morphisms $X \to Y$.

This problem is solved in characteristic 0 in the paper [BVS] for $M^1(X)$ and $M^{2d-1}(X)$. Their methods use resolution of singularities in an essential way, especially for $M^{2d-1}(X)$ and so require some modification is positive characteristic. We can prove the following:

Theorem (-M.). *For an algebraically closed field k of any characteristic, 1-motives $M^1(X)$, $M^1_c(X)$, $M^{2d-1}(X)$, $M^{2d-1}_c(X)$ can be defined which realize the corresponding cohomology groups, and they are unique up to isogeny of 1-motives.*

There are two problems left (which I hope to solve): functoriality for these 1-motives, and allowing the base field to be non-algebraically closed. We can also prove the following independence-of-ℓ statement:

Theorem (-M.). *Let X be a variety over a finite field k, and define $P^i_{(c)}(\ell, t) := \det(F - t1 | H^i_{(c)}(X_{\overline{k}}, \mathbb{Q}_\ell))$, the characteristic polynomial of the Frobenius element $F \in \mathrm{Gal}(\overline{k}/k)$ acting on the cohomology group $H^i_{(c)}(X_{\overline{k}}, \mathbb{Q}_\ell)$. Then $P^i_{(c)}(\ell, t)$ has integer coefficients independent of $\ell \neq p$ for $i = 0, 1, 2d-1, 2d$.*

We remark that when X is smooth and proper, independence is known for all i by [Del]. For arbitrary (singular, non-proper) X the only cases above which are new are $P^{2d-1}(\ell, t)$ and $P^{2d-1}_c(\ell, t)$.

References

[BVS] L. Barbieri-Viale and V. Srinivas, *Albanese and Picard 1-Motives*, Mémoire SMF **87**, Paris, 2001.

[Del] P. Deligne, *La conjecture de Weil: II*, Publ. Math. de l'I.H.E.S. **52** (1980), p. 137-252.