Towards a Tropical Proof of the Gieseker-Petri Theorem

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GIVEN a smooth projective curve X, we write $\mathcal{G}^r_d(X)$ for the variety parameterizing linear series of degree d and rank r on X. The nature of this variety for general curves is the central focus of two of the most celebrated theorems in modern algebraic geometry.

Brill-Noether Theorem. *[4] If X is a general curve of genus g, then dim $\mathcal{G}^r_d(X) = \rho = g - (r+1)(g-d+r)$.*

Gieseker-Petri Theorem. *[3] If X is a general curve, then $\mathcal{G}^r_d(X)$ is smooth.*

Recently, a team consisting of Cools, Draisma, Payne and Robeva provided an independent proof of the Brill-Noether Theorem using techniques from tropical geometry [2]. This semester, I led a small group of undergraduates towards a proof of the $r = 1$ case of the Gieseker-Petri Theorem using a similar approach.

Theorem. *The graph Γ_g does not admit a positive-rank divisor D such that $K_{\Gamma_g} - 2D$ is linearly equivalent to an effective divisor.*

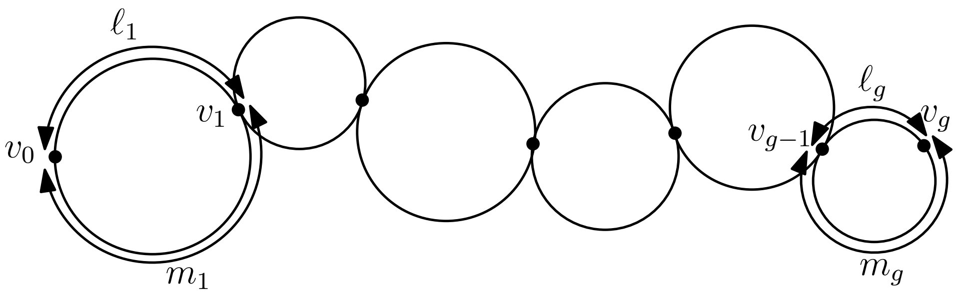

Figure 1: *The graph Γ_g from [2]*

Linear Series on Tropical Curves

A Divisor on a metric graph Γ is a formal sum of points of Γ. Given a continuous PL function ψ on Γ with integer slopes, we define $ord_p(\psi)$ to be the sum of the incoming slopes of ψ at the point $p \in \Gamma$. Two divisors D_1, D_2 on Γ are equivalent if $D_1 - D_2$ is of the form

$$div(\psi) := \sum_{p \in \Gamma} ord_p(\psi) p.$$

We say that a divisor D has *rank* r if r is the greatest integer such that $D - E$ is equivalent to an effective divisor for every effective divisor E of degree r.

Let R be a DVR with field of fractions K and residue field k, and let X be a smooth projective curve over K. Each point $p \in X(K)$ specializes to a smooth point of the special fiber X_k, and hence is associated to a well-defined vertex $\tau_*(p)$ of the dual graph Γ. Baker's Specialization Lemma then says that, for any divisor D on $X_{\bar{K}}$, $r(\tau_*(D)) \geq r(D)$ [1]. As a consequence, we see:

Theorem. *If the central fiber has dual graph Γ_g, then the theorem above implies that $\mathcal{G}^1_d(X)$ is smooth.*

Idea of the Proof

We work by induction on the genus g. The basic idea of the proof is to remove carefully chosen loops from the graph Γ_g as in the picture below. Notably, given a PL function ψ on the graph Γ_g, one can define a PL functions ψ' on Γ_{g-1} whose slope at every point other than the gluing vertex is identical to that of ψ.

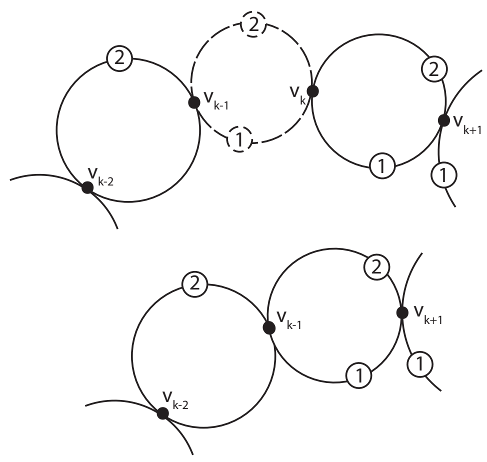

Figure 2: *Removing Loops*

References

[1] Matthew Baker, *Specialization of linear systems from curves to graphs*, Algebra Number Theory **2** (2008), no. 6, 613–653, With an appendix by Brian Conrad. MR 2448666 (2010a:14012)

[2] F. Cools, J. Draisma, S. Payne, and E. Robeva, *A tropical proof of the brill-noether theorem*, J. Reine Angew. Math.

[3] D. Gieseker, *Stable curves and special divisors: Petri's conjecture*, Invent. Math. **66** (1982), no. 2, 251–275. MR 656623 (83i:14024)

[4] Phillip Griffiths and Joseph Harris, *On the variety of special linear systems on a general algebraic curve*, Duke Math. J. **47** (1980), no. 1, 233–272. MR 563378 (81e:14033)