Quotient Models of Varieties over Complete Local Fields

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FIX a *complete local field* K with ring of integers $\mathcal{O}_K$, such that the residue field k of $\mathcal{O}_K$ is algebraically closed. One can pose the following question: How can one detect rational points of a smooth and projective K-variety X?

A special thing about varieties over local fields is that they come with models. A *model* $\mathcal{X}$ of a K-variety X is an S-variety, $S := Spec(\mathcal{O}_K)$, whose generic fiber is isomorphic to X. Note that there is a canonical map $\mathcal{X}(\mathcal{O}_K) \to X(K)$, which is a bijection if for example $\mathcal{X}$ is proper. Let $\mathcal{X}_k$ be the special fiber of $\mathcal{X}$. We get a specialization map $\psi : \mathcal{X}(\mathcal{O}_K) \to \mathcal{X}_k(k)$ by restricting $\mathcal{O}_K$-points to the special fiber.
If $x \in \mathcal{X}_k(k)$ is a regular point of $\mathcal{X}$, x is in the image of ψ if and only if it lies in the smooth locus of $\mathcal{X}$ over S. But given a singular point in $\mathcal{X}_k \subset \mathcal{X}$ one can not say whether there is a $\mathcal{O}_K$-point through it. To see this look at the following example:

Example. Let k be an algebraically closed field of $char(k) \neq 2$. Look at the complete local field $K = k((t))$. So $\mathcal{O}_K = k[\![t]\!]$. Let $X := V(tx_0x_1 - x_2^2) \subset \mathbb{P}^2_K$. X is a smooth projective K-variety $\mathcal{X} := V(tx_0x_1 - x_2^2) \subset \mathbb{P}^2_{\mathcal{O}_K}$ is a projective model of X. Note that $\mathcal{X}$ is singular for example in $P := (0, [1 : 0 : 0])$. Note that $U = Spec(k[\![t]\!][x_1, x_2]/(tx_1 - x_2^2))$ is an affine neighborhood of P, and $(x_1, x_2) \subset k[\![t]\!][x_1, x_2]/(tx_1 - x_2^2)$ defines a $\mathcal{O}_K$-point through P. Look at the smooth and projective $k[\![s]\!]$-scheme $\mathbb{P}^1_{k[\![s]\!]}$. Let $G = \mathbb{Z}/2\mathbb{Z}$ act on $\mathcal{Y}$ given by $g \in Aut(\mathbb{P}^1_{k[\![s]\!]})$ with $g((s, [y_0 : y_1])) = (-s, [-y_0 : y_1])$. Note that $\mathcal{X} = \mathbb{P}^1_{k[\![s]\!]}/G$.

A special kind of model of a K-variety X is a *weak Néron model*, which is a smooth model $\mathcal{X}$ of X with the property that the natural map from $\mathcal{X}(\mathcal{O}_L)$ to $X(K)$ is a bijection. Note that in this case X has a K-rational point if and only if the special fiber of $\mathcal{X}$ is not empty. One can construct out of a proper model of X a weak Néron model using the method of Néron smoothening. This works by blowing up singular points with sections through them. But a priori we do not know whether through a given singular point there is a section, so this method does not give us an explicit construction of a weak Néron model.

WE examine singular models of a special form. Fix a smooth projective K-variety X, and a tame Galois extension L/K. Then $G := Gal(L/K)$ acts on $X_L := X \times_{Spec(K)} Spec(L)$, and $X_L/G = X$. Fix a weak Néron model $\mathcal{Y}$ of X_L, such that the G-action on $X_L \subset \mathcal{Y}$ extends to an action on $\mathcal{Y}$ (one can show that such a $\mathcal{Y}$ always exists). Then $\mathcal{Y}/G$ is a model of X. In general $\mathcal{Y}/G$ will be singular. We will call such a model a *quotient model*. Note that the $\mathcal{X}$ examined in the example is a quotient model.

Theorem. *There is a weak Néron model $\mathcal{Z}$ of X endowed with a map to $\mathcal{Y}/G$, which is an isomorphism on the generic fiber, such that for every smooth S-scheme $\mathcal{V}$ a given dominant S-morphism $\Psi : \mathcal{V} \to \mathcal{Y}/G$ factors through $\mathcal{Z}$.*

Let the G-action on $\mathcal{Y}$ be given by $g \in Aut(\mathcal{Y})$ and that on $T = Spec(\mathcal{O}_L)$ by $g_T \in Aut(T)$. Then $\mathcal{Z}$ is given as a functor by

$$\begin{aligned} \mathcal{Z} : (Sch/S) &\to (Sets) \\ W &\mapsto \{\sigma \in Hom_T(W \times_S T, \mathcal{Y} \mid g\sigma \circ (id \times g_s)^{-1} = \sigma\} \end{aligned}$$

Using this explicit description of $\mathcal{Z}$ one can show for example the following corollary:

Corollary. *$\mathcal{Y}/G\ (\mathcal{O}_K) \neq \emptyset$ if and only if there exists a closed fixed point $y \in \mathcal{Y}$.*

ONE can use the results concerning quotient models to examine some motivic invariants of a K-variety X. Let $\mathcal{X}$ be a weak Néron model of a given K-variety X. The *motivic Serre invariant* $S(X)$ is the class of the special fiber of $\mathcal{X}$ in some quotient of $K_0(Var_{\mathbb{C}})/(\mathbb{L} - 1)$. Here $K_0(Var_{\mathbb{C}})$ is the *Grothendieck Ring of varieties*, generated by isomorphism classes $[U]$ of separated k-schemes of finite type and for every closed immersion $V \to U$ relations $[U] = [U \setminus V] + [V]$, with multiplication given by fiber product over k, and $\mathbb{L} = [\mathbb{A}^1_k]$.

Theorem. *Let X be a smooth and projective K-variety, let $\mathcal{Y}$ be a projective weak Néron model of X_L with a G-action as described above. Then*

$$S(X) = [\mathcal{Y}^G] \in K_0^{\mathcal{O}_K}(Var_{\mathbb{C}})/(\mathbb{L} - 1)$$

with $\mathcal{Y}^G \subset \mathcal{Y}$ the subscheme of fixed points.

The *rational volume* $s(x)$ is the Euler characteristic with proper support of the special fiber of $\mathcal{X}$ with coefficients in $\mathbb{Q}_l$.

Theorem. *Let X be a smooth and projective K-variety, and let L/K be a tame Galois extension, such that $Gal(L/K)$ is an l-group, l a prime. Then*

$$s(X) = s(X_L) \text{ mod } l$$

Note that if $S(X) \neq 0$ or $s(X) \neq 0$, then X has a K-rational point.