Bogomolov–Sommese vanishing on log canonical pairs

Patrick Graf
Albert–Ludwigs–Universität Freiburg
Mathematisches Institut

Advisor: Prof. Dr. Kebekus

Main result

The following well-known result is my starting point.

Theorem 1 (Bogomolov–Sommese vanishing, see [1])**.** *Let X be a complex projective manifold and $D \subset X$ a divisor with simple normal crossings. For any invertible subsheaf $\mathscr{L} \subset \Omega^p_X(\log D)$, we have $\kappa(\mathscr{L}) \leq p$, where $\kappa(\mathscr{L})$ denotes the Kodaira–Iitaka dimension of $\mathscr{L}$.*

Building on the Extension Theorem of Greb–Kebekus–Kovács–Peternell [2], I generalized this to the setting of reflexive differential forms on log canonical pairs as follows.

Theorem 2 (Bogomolov–Sommese vanishing on lc $\mathcal{C}$-pairs)**.** *Let (X, D') be a complex projective log canonical pair, and let $D \leq D'$ be a divisor such that (X, D) is a $\mathcal{C}$-pair. If $\mathscr{A} \subset \mathrm{Sym}^{[1]}_{\mathcal{C}} \Omega^p_X(\log D)$ is a Weil divisorial subsheaf, then $\kappa_{\mathcal{C}}(\mathscr{A}) \leq p$.*

A $\mathcal{C}$-pair is a pair (X, D) where all the coefficients of D are of the form $1 - 1/n$ for $n \in \mathbb{N} \cup \{\infty\}$. This notion was introduced by Campana under the name *orbifoldes géométriques*. The $\mathcal{C}$-Kodaira dimension $\kappa_{\mathcal{C}}$ of a Weil divisorial sheaf of differential forms on (X, D) is a natural generalization of the Kodaira dimension of a line bundle, which takes into account the fractional part of D.

Adjunction on dlt $\mathcal{C}$-pairs

In the course of the proof, I showed that on dlt $\mathcal{C}$-pairs, there is a version of the adjunction formula as well as a residue map for symmetric differential forms, and that these two are compatible with each other in the following sense.

Theorem 3 (Residues of symmetric differentials)**.** *Let (X, D) be a dlt $\mathcal{C}$-pair and $D_0 \subset \lfloor D \rfloor$ a component of the reduced boundary. Set $D_0^c := \mathrm{Diff}_{D_0}(D - D_0)$, such that $(K_X + D)|_{D_0} = K_{D_0} + D_0^c$. Then the pair (D_0, D_0^c) is also a dlt $\mathcal{C}$-pair, and for any integer $p \geq 1$, there is a map*

$$\mathrm{res}^k_{D_0} \colon \mathrm{Sym}^{[k]}_{\mathcal{C}} \Omega^p_X(\log D) \to \mathrm{Sym}^{[k]}_{\mathcal{C}} \Omega^{p-1}_{D_0}(\log D_0^c)$$

which on the snc locus of $(X, \lceil D \rceil)$ coincides with the k-th symmetric power of the usual residue map for snc pairs.

Corollary: A Kodaira–Akizuki–Nakano-type vanishing result

Corollary 4 (KAN-type vanishing)**.** *Let (X, D) be a complex projective log canonical pair of dimension n, $\mathscr{A}$ a Weil divisorial sheaf on X. Then*

$$H^n\big(X, \big(\Omega^{[p]}_X(\log\lfloor D\rfloor) \otimes \mathscr{A}\big)^{**}\big) = 0 \quad \text{and}$$
$$H^n\big(X, \Omega^p_X(\log\lfloor D\rfloor) \otimes \mathscr{A}\big) = 0$$

for $p \geq n - \kappa(\mathscr{A}) + 1$.

Idea of proof of Theorem 2

The basic idea is to pull back the sheaf $\mathscr{A}$ to a log resolution $(\tilde{X}, \tilde{D})$ of (X, D) and apply Theorem 1. By the Extension Theorem of [2], this should be possible. However, since pulling back is not functorial for Weil divisorial sheaves, the Kodaira dimension of $\mathscr{A}$ might drop in this process. Therefore we enlarge the pulled back sheaf by taking its saturation $\mathscr{B}$ in $\Omega^p_{\tilde{X}}(\log \tilde{D})$. We prove that sections of $\mathscr{A}^{[k]}$ extend to sections of $\mathscr{B}^{[k]}$.

A major issue is that we cannot really work on a log resolution, because it extracts too many divisors. Therefore we pass to a *minimal dlt model* (Z, D_Z) of (X, D). This is possible by the minimal model program as proved by BCHM. However, (Z, D_Z) is not an snc pair, which makes the proof rather involved. In particular, we have to use Theorem 3.

Sharpness of Theorem 2

Theorem 2 fails if one replaces log canonical by Du Bois singularities. A counterexample can be obtained as follows. Catanese constructed smooth projective surfaces S such that K_S is ample, but the Hodge numbers $h^{0,1}(S)$ and $h^{0,2}(S)$ are zero. Let X be the cone over such an S with respect to a sufficiently high pluricanonical embedding. Then X even has rational singularities, but the pullback of ω_S to X is a $\mathbb{Q}$-ample subsheaf of $\Omega^{[2]}_X$.

References

[1] Fedor A. Bogomolov, *Holomorphic tensors and vector bundles on projective varieties*, Math. USSR Izvestija **13** (1979), 499–555.

[2] Daniel Greb, Stefan Kebekus, Sándor J. Kovács, and Thomas Peternell, *Differential forms on log canonical spaces*, Publications Mathématiques de L'IHÉS **114** (2011), 1–83.