Principal Schottky Bundles Over a Compact Riemann Surface

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Moduli space of G-bundles

When studying the set of (stable or semistable) vector bundles, or more generally, principal G-bundles over a compact Riemann surface X with genus $g \geq 2$ we are looking for an algebraic variety which parametrizes the set of isomorphism classes of principal G-bundles. Since often principal G-bundles have many (nontrivial) automorphisms the moduli space may not be even an algebraic variety. In this way we have to make some restrictions in order to get a (coarse) moduli space of isomorphism classes of certain type of G-bundles over X. First, we denote by $\mathcal{M}_G^s\ (^{ss})$ the set of isomorphism classes of stable (semistable) principal G-bundles over X, this is a (coarse) moduli space and we have the following decomposition in connected components that are indexed by $\pi_1(G)$

$$\mathcal{M}_G^s = \coprod_{d \in \pi_1(G)} \mathcal{M}_G^{s,d}$$

where $\mathcal{M}_G^{s,d}$ is the set of isomorphism classes of stable G-bundles of topological type $d \in \pi_1(G)$. For any $g \geq 2$ and any $d \in \pi_1(G)$, by Ramanathan ([5], [4]), we have that each set $\mathcal{M}_G^{s,d}$ is nonempty and it is an open (and dense) subset of $\mathcal{M}_G^{ss,d}$.

Uniformization

The uniformization theorem says that any Riemann surface X of genus $g \geq 2$ can be written as a quotient of the upper half-plane by a Fuchsian group Γ, that is, $X \cong \mathbb{H}/\Gamma$ where $\Gamma \cong \pi_1(X)$. In the same way, the retrosection theorem for a compact Riemann surface X, $g \geq 2$, asserts that X can be written as Ω/Γ_s where Γ_s is a Schottky group with region of discontinuity $\Omega \subset \mathbb{CP}^1$. In particular, Γ_s is a free group F_g of rank g.

For vector bundles over a Riemann surface X there are similar notions. Narasimhan and Seshadri proved that every semistable vector bundle is induced by an unitary representation of the fundamental group. Ramanathan generalized Narasimhan and Seshadri's [3] result to principal G-bundles over a compact Riemann surface of genus $g \geq 2$ where G is a complex connected reductive algebraic group: each representation $\rho \in \mathrm{Hom}(\pi_1(X), G)$ induces a principal G-bundle $E_\rho = \widetilde{X} \times_\rho G$ where the following points

$$(\tilde{x}, g) \sim \left(\tilde{x}\gamma, \rho(\gamma)^{-1} \cdot g\right),\ \forall \gamma \in \pi_1(X)$$

are identified. In the case ρ unitary, E_ρ is semistable and if further ρ is irreducible, E_ρ is stable. In some way, the moduli space constructed by Ramanathan (and by Narasimhan and Seshadri) resembles the idea of "uniformization", that is, the moduli space of stable G-bundles over X can be seen as a quotient $\mathrm{Hom}^\sharp(\pi_1(X), G)/\!\!/G$ where $\mathrm{Hom}^\sharp(\pi_1(X), G)$ is the set of representations ρ such that E_ρ is a stable G-bundle (and a smooth point) of $\mathcal{M}_G^s$.

Schottky Bundles

Following Narasimhan and Seshadri's results and this ideas of uniformization, Florentino [1] introduced the notion of Schottky vector bundles over X. He proved that the map $\mathbf{V}.: \mathbb{G}_n^\sharp \to \mathcal{M}_n^s$ defined by $\mathbf{V}(\rho) = E_\rho$, where $\mathbb{G}_n^\sharp$ is the GIT quotient of simple Schottky representations is a local diffeomorphism in the neighborhood of unitary representations. Here the concept of Schottky representations means homomorphisms of $\pi_1(X)$ to $GL(n, \mathbb{C})$ with $\rho(\alpha_i) = 1$ for all $i = 1, \cdots, g$ where we consider $\pi_1(X)$ with the usual presentation $\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g | \prod_i [\alpha_i, \beta_i] = 1\}$.

In our work we generalize the notion of Schottky vector bundle to principal bundles. We defined **principal Schottky G-bundle** over X as $E_G \cong E_\rho$ where ρ is a Schottky representation, that is, if $\rho : \pi_1(X) \to G$ is a representation of the fundamental group of X in G such that $\rho(\alpha_i) \in Z(G)$ for all $i = 1, \cdots, g$ with $Z(G)$ the center of G. If E_G is a Schottky G-bundle then the adjoint bundle $\mathrm{Ad}(E_G) = E_G \times_{\mathrm{Ad}_\rho} \mathfrak{g}$ is a Schottky vector bundle with fibre $\mathfrak{g}$ induced by the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of G.

Let $\mathcal{S}$ be the set of Schottky representations, which is an algebraic subvariety of $\mathrm{Hom}(\pi_1(X), G)$ and let $\mathbb{S} = \mathcal{S}/\!\!/G$ be the corresponding GIT quotient. We proved that $\mathcal{S}$ is actually isomorphic to $\mathrm{Hom}(F_g, G \times Z(G))$ and since we are working with the free group F_g, with g generators, by Martin [2], $\mathbb{S} = \mathcal{S}/\!\!/G \cong \mathrm{Hom}(F_g, G \times Z(G))/\!\!/G \times Z(G)$ is irreducible and has dimension $(g-1)\dim G + (g+1)\dim Z(G)$.

Open Problem: Let $\mathbb{S}^\sharp = \{[\rho] \in \mathbb{S} : E_\rho \text{ is smooth and stable}\}$.

Is the map $\begin{array}{rccc} \mathbf{W}: & \mathbb{S}^\sharp & \to & \mathcal{M}_G^{smooth} \\ & [\rho] & \mapsto & [E_\rho] \end{array}$ surjective?

We proved that if we restrict to the case of unitary Schottky representations we have the following.

Main Theorem: The differential of the map

$$\begin{array}{rccc} \mathbf{W}: & \mathbb{S}^\sharp & \to & \mathcal{M}_G^{smooth} \\ & [\rho] & \mapsto & [E_\rho] \end{array}$$

at a point $[\rho] \in \mathbb{S}$ such that ρ is unitary and good (that is, stable and with $Stab(\rho) = Z(G \times Z(G))$), has maximal rank.

References

[1] C. Florentino, *Schottky uniformization and vector bundles over Riemann surface*, manuscripta math. **105** (2001), 69–83.

[2] B. Martin, *Restrictions of representations of surface group to a pair of free subgroups*, Journal of Algebra **225** (2000), 231–249.

[3] M.S. Narasimhan and C.S. Seshadri, *Holomorphic vector bundles on a compact Riemann surface*, Math. Ann. **155** (1964), 69–80.

[4] A. Ramanathan, *Stable principal bundles on a compact Riemann surface*, Math. Ann. **213** (1975), 129–152.

[5] ——, *Moduli for principal bundles over algebraic curves: I, ii*, Proc. Indian Acad. Sci. **106** (1996).